Power Solutions

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The Rules of Mastermind

MasterMind is a two-player code-breaking game. One of the players is the *code-setter*, and the other player acts as the *code-breaker*.

At the beginning of the game, the code-setter generates a secret code kept hidden from the code-breaker. The code consists of 4 pegs, each of which can be one of the six colors red, blue, green, yellow, orange, or violet. Throughout this round, these possible colors will be indicated by R, B, G, Y, O and V respectively. The code is allowed to contain repeated colors.

For example, RRBB, GYVO, RRRV, BYBO, and GGGG are all valid codes.

On the other hand, RRBK and GGGYB are examples of invalid codes that cannot be used in the game. The first code RRBK is not valid because it uses a label "K" not in our color list. The second code GGGYB is not a valid code because it consists of five colored pegs instead of four colored pegs.

The code-breaker will attempt to guess the secret code in a series of guess and feedback turns. Each turn proceeds as follows:

(i) The code-breaker guesses a code.

(ii) The code-maker will give a feedback (c, m) on the guess by telling the code-breaker the number c of pegs in the guess that had the *correct* color and position, and the number m of pegs in the guess that were correctly colored but *misplaced*.

These turns are repeated until the code-breaker correctly guesses the identity of secret-code. The objective of the code-breaker is to figure out the secret-code using as few guesses as possible.

For reference, below is a sample game, played out in its entirety, with the guesses and feedbacks presented for each turn.

Secret Code: RBYV

Turn	Guess	Correct	Misplaced
1	RRBB	1	1
2	GYRB	0	3
3	BRGO	0	2
4	RBVY	2	2
5	RBYV	4	0

In the above game, the code-setter picks RBYV as the secret code, and the code-breaker begins by guessing RRBB.

We observe that for this first guess, the first R peg is in the correct position. Moreover, the B peg is a color used in the secret code, but in the guess it is misplaced (it should go in the second slot instead of the third or fourth slot). So, the feedback that the code-breaker receives for this first guess is the ordered pair $(1, 1)$, corresponding to the information (**Correct**: 1, **Misplaced**: 1).

Note that even though there were 2 B pegs from guess the in the wrong position, the feedback was not $(1, 2)$. This is because only one of the B pegs could potentially be moved to fill in the second slot and match with the secret code. So, one of the B pegs in the guess was "misplaced", and the other was just incorrect - neither correct or misplaced.

Note also that when giving the feedback, the code-setter does **not** tell the code-breaker which pegs are correct and which are misplaced!

1 The Codes of MasterMind (7 pts)

1.1. How many possible secret codes can be generated using 4 pegs and 6 colors? **(2 pts)**

Answer

$\boxed{1296}$

Solution

There are 6 possible color choices for each of the 4 pegs.

Hence there are $6^4 = 1296$ valid codes.

1.2. Observe that for any possible feedback (c, m), we must have $c + m \leq 4$. How many ordered pairs of nonnegative integers (c, m) correspond to a valid feedback that might be given in the game of MasterMind? **(3 pts)**

Answer

$\boxed{14}$

Solution

If we have 0 pegs correct in our guess, then we can receive any of the five feedbacks of the form $(0, j)$ for $0 \leq j \leq 4$.

If we have 1 peg correct in our guess, then we can receive any of the four feedbacks of the form $(1, j)$ for $0 \leq j \leq 3$.

If we have 2 pegs correct in our guess, then we can receive any of the three feedbacks of the form $(2, j)$ for $0 \leq j \leq 2$.

If we have at least 3 pegs correct in our guess, the only possible feedbacks are $(3, 0)$ and $(4, 0)$.

Overall we then have $5 + 4 + 3 + 2 = 14$ possible feedbacks.

1.3. As it turns out, there is exactly one ordered pair of nonnegative integers (c, m) with $c + m = 4$ that is not attainable as a valid feedback in the game. What is this ordered pair and why is it not a valid feedback that could be received? **(2 pts)**

Answer
$\boxed{(3,1)}$
Solution
The feedback $(3, 1)$ is not attainable.

This is because if three of the four pegs are in the correct position, there is no way for the fourth remaining peg to be misplaced - either it is correct, or it is incorrect and not misplaced.

2 The Combinatorics of MasterMind (25 pts)

When playing the game, it can be helpful for the code-breaker to think about the set of possible secret codes that are consistent with all the feedback that they have received in the previous rounds. At the start of the game, before any guesses are made, the secret code could be any valid code. However, as the code-breaker makes more guesses, the number of possible secret codes diminishes.

For example, if the first guess is RRBB, and the feedback is $(1, 1)$, then the code-breaker now knows that RRRB is not the secret code. This is because if RRRB were the secret code, then the feedback for the guess RRBB would have been $(3, 0)$ instead.

2.1. If the first guess is GGGG, and the feedback is (0, 0), how many secret codes are still possible? **(2 pts)**

Answer
$\boxed{625}$
Solution
The feedback tells us that the secret code does not contain any G pegs.

It follows that there are $6 - 1 = 5$ choices for the color of each of the 4 pegs in the code, and thus there are $5^4 = 625$ secret codes still possible.

2.2. If the first guess is GGGY, and the feedback is $(1, 0)$, how many secret codes are still possible? **(3 pts)**

Answer
$\boxed{317}$
Solution
There are two cases to consider: either one of the G pegs is the single correct peg, or the Y peg is the single correct peg.

If one of the G pegs is correct, then there are 3 choices for which of the G pegs is correct. The remaining three pegs in the code cannot be G or Y pegs (since we had zero misplaced pegs), so there are $6 - 2 = 4$ possible colors for each of the remaining pegs.

It follows that there are $3 \cdot 4^3 = 192$ possible codes in this case.

If instead the Y peg is correct, then the first three pegs in the code cannot be G pegs. So there are $6 - 1 = 5$ possible colors for the first three pegs, yielding $5^3 = 125$ possible codes in this case.

Overall then there are $192 + 125 = 317$ possible codes left.

2.3. If the first guess is GGYY, and the feedback is (1, 0), how many secret codes are still possible?

(3 pts)

Answer

$\boxed{256}$

Solution

There are 4 possibilities for which peg in the guess is the correct peg.

Regardless of which peg is correct, since 0 pegs are misplaced, we know that the remaining three pegs in the secret code must all not be yellow or green.

It follows there are 4 possibilities for each of the three remaining pegs in each of the 4 cases.

Overall we then have $4 \cdot 4^3 = 256$ possible codes remaining.

2.4. Each of the following game boards shows the first few rounds of a different game of MasterMind. In each case find, with justification, how many secret codes are still possible for that game.

(a) **(5 pts)**

Turn	Guess	Correct	Misplaced
1	RRRR	0	0
2	BBBB	2	0
3	GGGG	0	0
4	YYYY	0	0
5	OOOO	1	0

Answer

$\boxed{12}$

Solution

The feedbacks tell us that the secret code consists of exactly 2 B pegs, exactly 1 O peg, and exactly 1 V peg. We get this last piece of information from the fact that the secret code has no R, G, or Y pegs.

All possible permutations of 2 B pegs, 1 O pegs, and 1 V peg will be consistent with the given feedback.

It follows that there are $\frac{4!}{2!1!1!} = 12$ possible codes remaining.

(b) **(5 pts)**

Turn	Guess	Correct	Misplaced
1	RRRR	1	0
2	BBBB	1	0
3	YYYY	1	0
4	GGGG	1	0
5	ROOO	0	1
6	BOOO	0	1
7	GOOO	0	1

Answer

$\boxed{6}$

Solution

The first four feedbacks tell us that the secret code consists of exactly one R peg, one B peg, one Y peg, and one G peg.

The last three feedbacks tell us that the secret code does not start with an R peg, B peg, or G peg. It follows that the code starts with a Y peg, and the remaining three pegs are some permutation of an R, a B, and a G.

Hence there are $3! = 6$ possible codes remaining.

(c) **(7 pts)**

Turn	Guess	Correct	Misplaced
1	RRBB	2	2
2	RBBR	2	2

Answer

$\boxed{2}$

Solution

The first feedback implies that the secret code has exactly 2 R pegs and 2 B pegs.

Moreover, in each guess exactly one R peg must be correct and exactly one B peg must be correct. If not, then two pegs of the same color would be correct, which would imply that the two pegs of the other color would also be correct.

Consider the first guess RRBB.

The first R and first B cannot be the two correct pegs, since the second guess RBBR agrees with RRBB in these positions, and thus one of RRBB or RBBR would have to be correct in this scenario.

A similar argument to the above shows that the second R and second B cannot be the two correct pegs.

It follows that either the first R and second B must be the two correct pegs, or the second R and first B must be the two correct pegs. In the first case the secret code RBRB is forced, and in the second case the secret code BRBR is forced.

Both of these codes are consistent with the given feedback.

Thus there are two possible codes.

3 The Strategy of MasterMind (45 pts)

In 1976, mathematician and computer scientist Donald Knuth found a strategy which allows the code-breaker in MasterMind to always correctly guess the secret code in 5 or fewer turns. The questions in this section will work towards proving the weaker result that there is a deterministic strategy for the code-breaker that can always guess the secret code in 12 or fewer turns.

One subproblem of guessing the secret code is to determine the color composition of the secret code. That is, the code-breaker would like to know for each color how many pegs of that color there are: how many R pegs are in the secret code, how many B pegs are in the secret code, etc.

3.1. Give a simple, deterministic (non-random) strategy that the codebreaker can use to always determine the color composition of the secret code in 6 or fewer guesses. **(2 pts)**

Answer
N/A

Solution
As suggested in problem 2.4, the codebreaker can just guess RRRR, OOOO, YYYY, GGGG, BBBB, and VVVV in that order.

The number correct in each of these guesses will indicate the number of pegs of that corresponding color, and allow the codebreaker to determine the color composition. Note that the number misplaced in each feedback will always be zero.

3.2. Give a deterministic strategy that can always determine the color composition of the secret code in 5 or fewer guesses. **(2 pts)**

Answer
N/A

Solution
The codebreaker can just guess RRRR, OOOO, YYYY, GGGG, and BBBB in that order.

The number correct in each of these guesses will indicate the number of pegs of that corresponding color. If the total number correct is less than 4, then the remaining pegs must be V pegs, since that is the only remaining color possible.

So these five guesses will allow the codebreaker to determine the color composition.

Suppose now that you are the code-breaker. Your friend was able to peek at the secret code, and they tell you that all 4 pegs in the code have the same color. Unfortunately, your friend forgot which color the pegs were. Luckily for you however, there are only 6 possibilities for the secret code: RRRR, OOOO, YYYY, GGGG, BBBB, or VVVV. So, as the code breaker, you could trivially guess the code in 6 or fewer guesses just by guessing all 6 of these possible codes in some order.

3.3. Give a deterministic strategy that you can use as the codebreaker to always guess the secret code in 5 or fewer guesses, given this information (hint: you do not have to guess all 4 pegs of the same color on the first turn!). **(5 pts)**

Answer
N/A

Solution
We can start by guessing RRBB.

If any pegs are listed as correct, then we guess RRRR and BBBB (one of which will be the secret code) and win in at most three guesses.

If we had 0 correct in our first guess, we then guess OOGG.

If any pegs are listed as correct, then we guess OOOO and GGGG (one of which will be the secret code) and win in at most four guesses.

If we had 0 correct on both the first and second guesses, we then guess YYYY and VVVV (one of which will be the secret code, since these are the only colors remaining) and win in at most four guesses.

Overall this strategy ensures that we win in fewer than 5 guesses.

3.4. Improve the strategy from problem 3.3 to get a strategy that you can use as the codebreaker to always guess the secret code in 3 or fewer guesses, given this information **(8 pts)**

Answer

N/A

Solution

We start by guessing RROY. If the feedback is (2,0), then the secret code is RRRR and we win in two guesses.

If the feedback is (1,0), the code is either OOOO or YYYY. We can guess these two codes and win in at most three guesses.

If the feedback is (0,0), then we guess GGGB on the second turn.

If the feedback of this second guess is (3,0), the code is GGGG and we win in three guesses.

If the feedback of this second guess is (1,0) the code is BBBB, and we win in three guesses.

Finally, if the feedback is (0,0), the code is VVVV (since this is the only color remaining), and we in three guesses in this case too.

Hence this strategy lets us win in at most three guesses.

We move on to considering strategies for the code-breaker in the general MasterMind game, where we aren't guaranteed that all 4 pegs are of the same color. In coming up with strategies in the general case, it might help to use the ideas from the previous problems about color composition.

3.5. For the standard Mastermind game, give a deterministic strategy for the codebreaker that can always guess the secret code in strictly less than 1300 turns. **(3 pts)**

Answer

N/A

Solution

From the solution to problem 1.1, we know that there are fewer than 1300 valid codes in the game. So, the codebreaker just go through and guess all the valid codes in some arbitrary order and ensure victory in fewer than 1300 turns.

3.6. Give a deterministic strategy for the codebreaker that can always guess the secret code in 30 or fewer guesses (hint: $30 = 6 + 4!$, there are 6 possible colors and 4 pegs in the code).

(10 pts)

Answer

N/A

Solution

Using the strategy from the solution to problem 3.1, the codebreaker can determine the color composition of the secret code in 6 guesses.

Once the codebreaker knows the color composition, there are at most $4! = 24$ possible codes left (corresponding to the arrangements of the given colors of pegs).

The codebreaker can just go through and guess all of these possible combinations, and guarantee victory in at most $6 + 24 = 30$ turns.

3.7. Improve the strategy from problem 3.6 to get a strategy that guarantees the code-breaker can always guess the secret code in 20 or fewer guesses. **(7 pts)**

Answer

N/A

Solution

Using the strategy from the solution to problem 3.1, the codebreaker can determine the color composition of the secret code in 6 guesses.

Once the codebreaker knows the color composition is C, D, E, F for some colors C, D, E, F, they can pick some color X not in the secret code.

They can then guess CXXX and DXXX. At the very least, this will limit the number of possibilities for the first peg to having one of 2 colors. The remaining three pegs can be arranged in at most $3! = 6$ ways.

It follows that after the above two guesses there are at most $2 \cdot 3! = 12$ possible codes left

The codebreaker can just go through and guess all of these possible combinations, and guarantee victory in at most $6 + 2 + 12 = 20$ turns.

3.8. Improve the strategy from problem 3.6 to get a strategy that guarantees the code-breaker can always guess the secret code in 12 or fewer guesses. **(8 pts)**

Answer

N/A

Solution

Using the strategy from the solution to problem 3.2, the codebreaker can determine the color composition of the secret code in 5 guesses.

Once the codebreaker knows the color composition is C, D, E, F for some colors C, D, E, F, they can pick some color X not in the secret code.

They can then guess CXXX, DXXX, and EXXX in that order. Using these 3 guesses the codebreaker can determine the color of the first peg (this is clear if one of the feedbacks returns 1 correct, and all the feedbacks return 0 correct, the first peg must have color F).

Without loss of generality suppose the first peg has color C. Then the codebreaker can guess XDXX and XEXX. Using these 2 gusses the codebreaker can determine the color of the second peg.

Finally, guessing XXFX allows the codebreaker to determine the color of the third peg (and hence also the final peg, since the codebreaker knows the color composition of the code). Then, the codebreaker uses one more guess to guess the correct secret code (which by now they have figured out).

Overall this allows the codebreaker to win in at most $5 + 3 + 2 + 1 + 1 = 12$ guesses as desired.

The codebreaker can just go through and guess all of these possible combinations, and guarantee victory in at most $6 + 24 = 30$ turns.

4 The Bounds of MasterMind (13 pts)

In this final section we will prove that there is no strategy for the code-breaker that guarantees that they can always guess the secret code in 3 or fewer guesses.

At the beginning of the game, there are N possible secret codes the code-setter could set (where N is the answer to problem 1.1), and there are M possible feedbacks (where M is the answer to problem 1.2). After the code-breaker makes the first guess, but before they receive the first feedback, we can partition the set of N possible codes into M sets, based on which feedback the code-breaker would see if a particular code was the secret code.

4.1. Show that no matter what the first guess is, there exists some feedback such that there will still be at least 80 possible secret codes after that feedback is received. **(3 pts)**

Answer

$\boxed{\text{N/A}}$

Solution

From the results of the previous problems, we know that $N = 1296$ and $M = 14$.

Then by the pigeonhole principle, there exists a feedback so that the number of possible codes after receiving that feedback is at least

$$\left\lceil \frac{N}{M} \right\rceil = \left\lceil \frac{1296}{14} \right\rceil = 93 > 80$$

as desired.

4.2. Using the result of problem 4.1, show that there cannot be a deterministic strategy that always guesses the secret code in 2 or fewer guesses. **(3 pts)**

Answer

$\boxed{\text{N/A}}$

Solution

From the result of problem 4.1, regardless of the first guess there is more than one possible code left. Thus, it is impossible for the codebreaker to guarantee that there second guess will be correct, and hence there is no strategy that will ensure victory in at most 2 guesses.

4.3. Extend the reasoning from problems 4.1 and 4.2 to show that there cannot be a deterministic strategy that always guesses the secret code in 3 or fewer guesses. **(7 pts)**

Answer

$\boxed{\text{N/A}}$

Solution

Extending the reasoning from the solutions to 4.1 and 4.2, after the second guess, there exists some feedback pair that leaves at least

$$\left\lceil \frac{80}{14} \right\rceil = 6$$

possible codes.

Hence, for any strategy, there will always be two feedbacks for the first two turns that leaves the codebreaker with more than one possible code going into the 3rd turn. It is impossible then for the codebreaker to guarantee that their third guess will be correct.

Hence there is no strategy that ensures victory in at most 3 guesses.