Individual Solutions

November 19, 2017

1. A dog on a 10 meter long leash is tied to a 10 meter long, infinitely thin section of fence. What is the minimum area over which the dog will be able to roam freely on the leash, given that we can fix the position of the leash anywhere on the fence?

   Proposed by Oriel Humes

   Answer
   $\boxed{75\pi}$

   Solution
   Suppose the leash is fixed to a point on the fence which is a distance x away from the left end of the fence.
   The dog can always roam in a semicircle of radius 10 meters if she stays on the same side of the fence the leash is tied to. She can also roam around the corners of the fence, which adds an additional roaming region of two semicircles of radius x and $10 - x$.
   Hence the total area the dog can roam is

   $$\frac{\pi}{2}\left(10^2 + x^2 + (10-x)^2\right).$$

   By the QM-AM inequality, we have

   $$x^2 + (10-x)^2 \geq \left(\frac{x + (10-x)}{2}\right)^2 = 5^2$$

   with equality exactly when $x = 5$.
   Thus the minimum area the dog can roam is

   $$\frac{\pi}{2}\left(10^2 + 5^2\right) = 75\pi.$$

2. Suppose that the equation

   $$\begin{array}{cccccc} & \underline{C} & \underline{H} & \underline{M} & \underline{M} & \underline{C} \\ + & & \underline{H} & \underline{M} & \underline{M} & \underline{T} \\ \hline & \underline{P} & \underline{U} & \underline{M} & \underline{A} & \underline{C} \end{array}$$

   holds true, where each letter represents a single nonnegative digit, and **distinct letters represent different digits** (so that $\underline{C}\ \underline{H}\ \underline{M}\ \underline{M}\ \underline{C}$ and $\underline{P}\ \underline{U}\ \underline{M}\ \underline{A}\ \underline{C}$ are both five digit positive integers, and the number $\underline{H}\ \underline{M}\ \underline{M}\ \underline{T}$ is a four digit positive integer).
   What is the largest possible value of the five digit positive integer $\underline{C}\ \underline{H}\ \underline{M}\ \underline{M}\ \underline{C}$?
   Proposed by Shyan Akmal

Answer

$\boxed{65996}$

Solution

By looking at the least significant digit of the terms in the given sum, we see that $\mathsf{T} = 0$ is forced.

By looking at the most significant digit of the terms in the given sum, and using the fact that $\mathsf{C} \neq \mathsf{P}$, we see that $\mathsf{P} = \mathsf{C} + 1$ is forced.

The above two observations imply that the equation

$$\begin{array}{cccc} & \underline{\mathsf{H}} & \underline{\mathsf{M}} & \underline{\mathsf{M}} \\ + & \underline{\mathsf{H}} & \underline{\mathsf{M}} & \underline{\mathsf{M}} \\ \hline \underline{1} & \underline{\mathsf{U}} & \underline{\mathsf{M}} & \underline{\mathsf{A}} \end{array}$$

must hold true as well.

The above equation is equivalent to

$$200\mathsf{H} + 22\mathsf{M} = 1000 + 100\mathsf{U} + \mathsf{A}$$

which after canceling Ms yields

$$200\mathsf{H} + 12\mathsf{M} = 1000 + 100\mathsf{U} + \mathsf{A}.$$

Since the left hand side is a multiple of 4 and $1000 + 100\mathsf{U}$ is a multiple of 4, the above equation implies that A is an integer multiple of 4 as well.

Since A is a digit, it follows that $\mathsf{A} = 4\mathsf{A}'$ for some $A' \in \{1, 2\}$. If we substitute this into the above equation, we get

$$50\mathsf{H} + 3\mathsf{M} = 250 + 25\mathsf{U} + \mathsf{A}'.$$

We can then rearrange the above equation to get

$$25\left(10 + \mathsf{U} - 2\mathsf{H}\right) = 3\mathsf{M} - \mathsf{A}'.$$

The left hand is divisible by 25, so the right hand side is a multiple of 25 as well. Since $\mathsf{A}' \in \{1, 2\}$ and M is a digit, it follows that the right hand side must equal 25.

This forces $\mathsf{A}' = 2$ and $\mathsf{M} = 9$. Substituting this into the above equation yields

$$2\mathsf{H} - \mathsf{U} = 9.$$

This last equation only has solutions

$$(\mathsf{H}, \mathsf{U}) \in \{(5, 1), (6, 3), (7, 5), (8, 7), (9, 9)\}.$$

Since we want to maximize $\underline{\mathsf{C}}\ \underline{\mathsf{H}}\ \underline{\mathsf{M}}\ \underline{\mathsf{M}}\ \underline{\mathsf{C}}$, we want to maximize the value of $\underline{\mathsf{C}}$. To do this we just take $\mathsf{C} = 6$ to be as large as possible, which then results in $\mathsf{H} = 5$, which yields

$$\underline{\mathsf{C}}\,\underline{\mathsf{H}}\,\underline{\mathsf{M}}\,\underline{\mathsf{M}}\,\underline{\mathsf{C}} = 65996.$$

3. Square $ABCD$ has side length 4, and E is a point on segment BC such that $CE = 1$. Let $\mathcal{C}_1$ be the circle tangent to segments AB, BE, and EA, and $\mathcal{C}_2$ be the circle tangent to segments CD, DA, and AE.

   What is the sum of the radii of circles $\mathcal{C}_1$ and $\mathcal{C}_2$?

   Proposed by Shyan Akmal

   Answer
   $\boxed{7/3}$

   Solution
   Let X denote the intersection of line AE with line CD.
   We first observe that ABE is a 3-4-5 triangle. Thus it has inradius $\frac{3+4-5}{2} = 1$.
   Then because $\mathcal{C}_1$ is the incircle of ABE, it has radius 1.
   Now because AB and DX are parallel lines, we have $\angle DXA = \angle EAB$. Since XDA and ABE are both right triangles, it follows that $XDA \sim ABE$ are similar.
   The ratio of the similtude from XDA to ABE is $DA/BE = 4/3$.
   Then because $\mathcal{C}_2$ is the incircle of XDA, it has radius $4/3 \cdot 1 = 4/3$.
   Thus the sum of the radii of $\mathcal{C}_1$ and $\mathcal{C}_2$ is $7/3$.

4. A finite set S of points in the plane is called *tri-separable* if for every subset $A \subseteq S$ of the points in the given set, we can find a triangle $\mathcal{T}$ such that

   (i) every point of A is inside $\mathcal{T}$, and
   (ii) every point of S that is **not** in A is outside $\mathcal{T}$.

   What is the smallest positive integer n such that no set of n distinct points is tri-separable?

   Proposed by Albert Tseng

   Answer
   $\boxed{8}$

   Solution
   We can verify by inspection that if we take S to be the vertices of a regular heptagon then S is tri-separable. Thus $n > 7$.
   Now, take an arbitrary set S of 8 points in the plane and consider the convex hull Ω of S.
   If one of the points of S is not on Ω, we can consider the subset A of all the points on Ω. It is clear that there is no triangle which has A in its interior yet has $S \setminus A$ in its exterior (since a triangle is itself a convex figure). Hence in this case S is not tri-separable.
   Otherwise, all points in S are on Ω. In this case take A to be "every other" point on Ω, starting with some arbitrary point in the set and moving anticlockwise around Ω.
   If we appeal to the fact that triangles are convex and only have three sides, we can show that there is no way for a triangle to contain exactly the points of A in its interior in our above setup.
   Hence in this case as well S is not tri-separable. Thus $n = 8$.

5. The unit 100-dimensional hypercube $\mathcal{H}$ is the set of points $(x_1, x_2, \ldots, x_{100})$ in $\mathbb{R}^{100}$ such that $x_i \in \{0, 1\}$ for $i = 1, 2, \ldots, 100$. We say that the *center* of $\mathcal{H}$ is the point

$$\left(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}\right)$$

in $\mathbb{R}^{100}$, all of whose coordinates are equal to $1/2$.

For any point $P \in \mathbb{R}^{100}$ and positive real number r, the *hypersphere* centered at P with radius r is defined to be the set of all points in $\mathbb{R}^{100}$ that are a distance r away from P.

Suppose we place hyperspheres of radius $1/2$ at each of the vertices of the 100-dimensional unit hypercube $\mathcal{H}$. What is the smallest real number R, such that a hypersphere of radius R placed at the center of $\mathcal{H}$ will intersect the hyperspheres at the corners of $\mathcal{H}$?

Proposed by Oriel Humes

Answer

$\boxed{9/2}$

Solution

By the Pythagorean theorem, the longest diagonal of $\mathcal{H}$ has length $\sqrt{100} = 10$.

Hence the smallest hypersphere placed at the center of $\mathcal{H}$ which intersects the hyperspheres in the corners must have diameter $10 - 2(1/2) = 9$ (this is just the length of the longest diagonal minus the radii of the two hyperspheres at opposite corners of $\mathcal{H}$).

It follows that the smallest possible radius R is $9/2$.

6. Greg has a 9×9 grid of unit squares. In each square of the grid, he writes down a single **nonzero** digit.

Let N be the number of ways Greg can write down these digits, so that each of the nine nine-digit numbers formed by the rows of the grid (reading the digits in a row left to right) and each of the nine nine-digit numbers formed by the columns (reading the digits in a column top to bottom) are multiples of 3.

What is the number of positive integer divisors of N?

Proposed by Shyan Akmal

Answer

$\boxed{146}$

Solution

In the following solution, we appeal to the fact that an integer is divisible by 3 if and only if the sum of its digits is divisible by 3.

Consider the top left 8×8 subgrid of squares. Greg can fill these $8^2 = 64$ entries arbitrarily with nonzero digits. There are $9^{64} = 3^{128}$ ways to do this.

Once he does this, for each row, there is unique value modulo 3 that can be placed in the final entry of that row to ensure that the number in that row (reading left to right) is divisible by 3. The same reasoning holds for each column of the grid.

Since each residue modulo 3 corresponds to three positive digits, it follows that there are $3^{2 \cdot 9 - 1} = 3^{17}$ ways to fill in the bottom row and rightmost column of the grid.

Overall then there are $N = 3^{128+17} = 3^{145}$ ways to fill in the grid. Thus the number of positive integer divisors of N is just $145 + 1 = 146$.

Note: In the above solution, we technically need to be careful about how we fill in the bottom rightmost entry of the grid, since it's value affects both the bottom row number and the rightmost column number, and it might not be clear how we can ensure that both of these numbers will simultaneously be multiples of 3.

However, it turns out that our choice of that value (uniquely determined modulo 3) still always works out, and we can prove this by considering the sum of all of the entries in the grid (first by rows, then by columns).

7. Find the largest positive integer n for which there exists positive integers x, y, and z satisfying

$$n \cdot \gcd(x, y, z) = \gcd(x + 2y, y + 2z, z + 2x).$$

Proposed by Vinayak Kumar & Gideon Leeper

Answer

$\boxed{9}$

Solution

Let $d = \gcd(x, y, z)$.

Then we may write $x = da, y = db, y = dc$ for some integers a, b, c satisfying $\gcd(a, b, c) = 1$.

With this representation, the given equation simplifies to

$$n = \gcd(a + 2b, b + 2c, c + 2a).$$

Since n divides all the arguments on the right hand side, n must also divide

$$(a + 2b) + 2(b + 2c) + 4(c + 2a) = 9a + 4(b + 2c).$$

Then because n divides $b + 2c$, the above equation implies that n also divides $9a$.

Similar reasoning shows that n divides $9b$ and $9c$.

It follows that n divides

$$\gcd(9a, 9b, 9c) = 9 \cdot \gcd(a, b, c) = 9.$$

Hence $n \le 9$. We can show that $n = 9$ is achievable by taking $(x, y, z) = (1, 4, 7)$, since in this case

$$\gcd(x, y, z) = 1$$

and

$$\gcd(x + 2y, y + 2z, z + 2x) = \gcd(9, 18, 9) = 9$$

so $n = 9$ is our final answer.

8. Suppose $ABCDEFGH$ is a cube of side length 1, one of whose faces is the unit square $ABCD$. Point X is the center of square $ABCD$, and P and Q are two other points allowed to range on the surface of cube $ABCDEFHG$. Find the largest possible volume of tetrahedron $AXPQ$.

Proposed by Adam Busis

Answer

$\boxed{1/6}$

Solution

Fix point P somewhere on the cube. Then because the cube is convex, the volume of $AXPQ$ will be maximized (for this fixed position of P) when Q is at one of the vertices of the cube.

The same reasoning holds if we fix Q and instead try to maximize the volume by moving P. It follows that to maximize the volume of $AXPQ$, we can set P and Q to both be vertices of the cube.

From here we can just go through different cases for the placement of P and Q. After we check these cases, the maximum possible volume ends up being $1/6$.

9. Deep writes down the numbers $1, 2, 3, \ldots, 8$ on a blackboard. Each minute after writing down the numbers, he uniformly at random picks some number m written on the blackboard, erases that number from the blackboard, and increases the values of all the other numbers on the blackboard by m. After seven minutes, Deep is left with only one number on the black board.

   What is the expected value of the number Deep ends up with after seven minutes?

   Proposed by Shyan Akmal

Answer

$\boxed{576}$

Solution

Imagine that the numbers on the blackboard occupy 8 slots.

Initially, the number k is in the k^{th} slot of the blackboard for $k = 1, 2, \ldots, 8$. When we increase the values of numbers on the blackboard, they stay in the same slot.

For $k = 1, 2, \ldots, 8$, let a_k denote the slot we erase from in the k^{th} minute. For example, a_1 is the slot number Deep first erased from, and a_8 is the slot number Deep erases from last.

We can check that the final number Deep ends up with is equal to

$$a_8 + a_7 + 2a_6 + 2^2 a_5 + \ldots + 2^6 a_1.$$

Since each a_i is equally likely to be any of the values from $\{1, 2, \ldots, 8\}$, by linearity of expectation we find that the expected value of the number Deep ends up with is

$$\left(\frac{1 + 2 + \ldots + 8}{8}\right)\left(1 + 1 + 2 + 2^2 + \ldots + 2^6\right) = \frac{9}{2} \cdot 2^7 = 9 \cdot 64 = 576.$$

Note: It also possible to solve this problem recursively, and using an induction argument show that if we start off with a list S of n numbers on the board, then the expected sum we end up with in the end is

$$\frac{2^{n-1}}{n} \cdot \sum_{s \in S} s.$$

10. Find the number of ordered tuples $(x_1, x_2, x_3, x_4, x_5)$ of positive integers such that $x_k \leq 6$ for each index $k = 1, 2, \ldots, 5$, and the sum

$$x_1 + x_2 + \cdots + x_5$$

is 1 more than an integer multiple of 7.

Proposed by Gideon Leeper & Felix Weilacher

Answer

$\boxed{1111}$

Solution

Let r_n denote the number of ordered tuples $(x_1, x_2, \ldots, x_n)$ of residues modulo 7 such that

$$x_1 + x_2 + \cdots + x_n \equiv 0 \pmod{7}.$$

Let s_n denote the number of ordered tuples $(x_1, x_2, \ldots, x_n)$ of residues modulo 7 such that

$$x_1 + x_2 + \cdots + x_n \equiv 1 \pmod{7}.$$

Note that for this problem, we just want to compute s_5.

It turns out that for each nonzero residue k modulo 7, the number s_n also counts the number of ordered tuples $(x_1, x_2, \ldots, x_n)$ of residues modulo 7 such that

$$x_1 + x_2 + \ldots + x_n \equiv k \pmod{7}.$$

This is because the map

$$(a_1, a_2, \ldots, a_n) \mapsto (ka_1, ka_2, \ldots, ka_n) \pmod{p}$$

is a bijection from the solution set of

$$x_1 + x_2 + \ldots + x_n \equiv 1 \pmod{7}$$

ot the solution set of

$$x_1 + x_2 + \ldots + x_n \equiv k \pmod{7}$$

for each nonzero residue modulo 7.

Then by considering possible values for x_{n+1}, the last component of a solution to any one of these congruences, we get the recurrences $r_{n+1} = 6s_n$ and $s_{n+1} = r_n + 5s_n$. If we substitute the first recurrence into the second recurrence we get

$$s_{n+1} = 5s_n + 6s_{n-1} \qquad (n \geq 1).$$

We can solve this linear recurrence along with the initial conditions $s_0 = 0$ and $s_1 = 1$ to find that

$$s_n = \frac{6^n - (-1)^n}{7}.$$

It follows that the answer to the problem is

$$s_5 = \frac{6^5+1}{7} = 1111.$$

Note: It is also possible to solve this problem with complex numbers or generating functions.

11. The equation

$$\left(x-\sqrt[3]{13}\right)\left(x-\sqrt[3]{53}\right)\left(x-\sqrt[3]{103}\right)=\frac{1}{3}$$

has three distinct real solutions $r, s,$ and t for x.

Calculate the value of

$$r^3+s^3+t^3.$$

Proposed by Shyan Akmal

Answer

$\boxed{170}$

Solution

Consider the polynomial $f(x)=\left(x-\sqrt[3]{13}\right)\left(x-\sqrt[3]{53}\right)\left(x-\sqrt[3]{103}\right)$.

Let $\alpha=\sqrt[3]{13}, \beta=\sqrt[3]{53}$, and $\gamma=\sqrt[3]{103}$ be the roots of f. Note that $r, s,$ and t are roots of

$$f(x)-1/3.$$

By Vieta's formulas we then have

$$r+s+t=\alpha+\beta+\gamma$$
$$rs+st+tr=\alpha\beta+\beta\gamma+\gamma\alpha$$
$$rst=\alpha\beta\gamma+\frac{1}{3}.$$

From the factorization (valid for all complex numbers u, v, w)

$$u^3+v^3+w^3-3uvw=(u+v+w)\left((u+v+w)^2-3(uv+vw+wu)\right)$$

and the first two equations from above we know that

$$r^3+s^3+t^3-3rst=\alpha^3+\beta^3+\gamma^3-3\alpha\beta\gamma.$$

It follows then, using the third equation from above, that

$$r^3+s^3+t^3=\left(\alpha^3+\beta^3+\gamma^3\right)+3(rst-\alpha\beta\gamma)=(13+53+103)+3\cdot\frac{1}{3}=170.$$

Note: More generally, if $f(x)$ is a polynomial of degree $n\ge 1$ with roots $\alpha_1, \alpha_2, \ldots, \alpha_n$ and $f(x)-c$ has roots $r_1, r_2, \ldots, r_n$ (where c is some complex number), then using Newton's sums one can show that for $\ell=1,2,\ldots,n-1$ we have

$$r_1^\ell + r_2^\ell + \ldots + r_n^\ell = \alpha_1^\ell + \alpha_2^\ell + \ldots + \alpha_n^\ell.$$

Additionally, we have

$$r_1^n + r_2^n + \ldots + r_n^n = \alpha_1^n + \alpha_2^n + \ldots + \alpha_n^n + (-1)^{n+1} nc.$$

12. Suppose $a, b,$ and c are real numbers such that

$$\frac{ac}{a+b} + \frac{ba}{b+c} + \frac{cb}{c+a} = -9$$

and

$$\frac{bc}{a+b} + \frac{ca}{b+c} + \frac{ab}{c+a} = 10.$$

Compute the value of

$$\frac{b}{a+b} + \frac{c}{b+c} + \frac{a}{c+a}.$$

Proposed by Shyan Akmal

Answer
$\boxed{11}$

Solution
Adding the two given equations and collecting terms with the same denominator yields

$$a + b + c = 1. \qquad (\star)$$

Subtracting the the two given equations yields

$$\sum_{\text{cyc}} \frac{c(a-b)}{a+b} = -19. \qquad (*)$$

Let us call the value we want to find V, and define the natural complementary value

$$U = \frac{a}{a+b} + \frac{b}{b+c} + \frac{c}{c+a}.$$

First off, we have

$$U + V = 3$$

just by collecting terms with the same denominator.
We also have

$$U - V = \sum_{\text{cyc}} \frac{a-b}{a+b}.$$

In fact, if we multiply $(\star)$ to the above equation and simplify we find that

$$\begin{aligned}U-V &= \sum_{\text{cyc}}\left((a+b+c)\cdot\frac{a-b}{a+b}\right)\\ &= \sum_{\text{cyc}}\left((a-b)+\frac{c(a-b)}{a+b}\right)\\ &= \sum_{\text{cyc}}(a-b)+\sum_{\text{cyc}}\frac{c(a-b)}{a+b}\\ &= -19\end{aligned}$$

where in the last step we used $(*)$.
With this, we can then compute the answer as

$$V=\frac{(U+V)-(U-V)}{2}=\frac{3-(-19)}{2}=11.$$

13. The complex numbers w and z satisfy the equations $|w|=5, |z|=13$, and

$$52w-20z=3(4+7i).$$

Find the value of the product wz.

Proposed by Evan Liang

Answer

$\boxed{33-56i}$

Solution

Notice that

$$\begin{aligned}52z_1-20z_2 &= 52z_1\frac{z_2\overline{z_2}}{13^2}-20z_2\frac{z_1\overline{z_1}}{5^2}\\ &= 4z_1z_2\left(\frac{\overline{z_2}}{13}-\frac{\overline{z_1}}{5}\right)\\ &= z_1z_2\left(\frac{20\overline{z_2}-52\overline{z_1}}{65}\right).\end{aligned}$$

It follows that

$$\begin{aligned}z_1z_2 &= -65\frac{52z_1-20z_2}{52\overline{z_1}-20\overline{z_2}}\\ &= -65\frac{52z_1-20z_2}{\overline{52z_1-20z_2}}\\ &= -65\frac{3(4+7i)}{3(4-7i)}\\ &= -65\frac{(4+7i)^2}{4^2+7^2}\\ &= -(4+7i)^2\\ &= 33-56i\end{aligned}$$

is the value of the product of the two complex numbers.

14. For $i = 1, 2, 3, 4$, we choose a real number x_i uniformly at random from the closed interval $[0, i]$. What is the probability that $x_1 < x_2 < x_3 < x_4$?

Proposed by Adam Busis

Answer

$\boxed{125/576}$

Solution

We can break into cases based on which unit interval $[n-1, n]$ each of the x_i is in.

For example, if $x_1, x_2 \in [0, 1]$ and $x_3, x_4 \in [1, 2]$ then the probability that x_1, x_2, x_3, x_4 are strictly increasing is $(1/2)(1/2) = 1/4$. If we use (a_1, a_2, a_3, a_4) to denote the case where $x_i \in [a_i - 1, a_i]$ then the probabilities of each case are:

- $1/24$ for $(1, 1, 1, 1)$
- $1/6$ for $(1, 1, 1, 2)$, $(1, 1, 1, 3)$, $(1, 1, 1, 4)$, $(1, 2, 2, 2)$
- $1/4$ for $(1, 1, 2, 2)$, $(1, 1, 3, 3)$
- $1/2$ for $(1, 1, 2, 3)$, $(1, 1, 2, 4)$, $(1, 1, 3, 4)$, $(1, 2, 2, 3)$, $(1, 2, 2, 4)$, $(1, 2, 3, 3)$
- 1 for $(1, 2, 3, 4)$

Adding these up and multiplying by $1/24$ (the probability for each case) gives the answer.

Note: Alternate solutions to the general case of this problem (where we look at n randomly chosen numbers instead of just 4, where n is some positive integer) can be found a as the solutions to **HMMT February C10**.

15. The terms of the infinite sequence of rational numbers $a_0, a_1, a_2, \ldots$ satisfy the equation

$$a_{n+1} + a_{n-2} = a_n a_{n-1}$$

for all integers $n \geq 2$.

Moreover, the values of the initial terms of the sequence are

$$a_0 = \frac{5}{2}, a_1 = 2, \text{ and } a_2 = \frac{5}{2}.$$

Call a **nonnegative** integer m *lucky* if when we write

$$a_m = \frac{p}{q}$$

for some relatively prime positive integers p and q, the integer $p + q$ is divisible by 13.

What is the 101^{st} smallest lucky number?

Proposed by Shyan Akmal

Answer

$\boxed{1207}$

Solution

Using induction, we can show that

$$a_n = 2^{F_{n-1}} + 2^{-F_{n-1}}$$

for all positive integers n, where F_n denotes the n^{th} term in the Fibonacci sequence

$$F_{-1} = 1, F_0 = 0, F_1 = 1, F_2 = 2, \ldots$$

So for any nonnegative integer m, when we write $a_m = p/q$ for relatively prime positive integers p and q, we must have $p = 2^{2F_{m-1}} + 1$ and $q = 2^{F_{m-1}}$.

Thus m is lucky if and only if $13 \mid 2^{2F_{m-1}} + 2^{F_{m-1}} + 1$.

From here there are multiple ways to finish.

One approach is to note that the above divisibility condition just means that $2^{F_{m-1}}$ is a root of $x^2 + x + 1$ modulo 13. Since $13 = 3^2 + 3 + 1$, one root of $x^2 + x + 1$ modulo 13 is 3. By Vieta's formulas, the other root is $-1 - 3 = -4 \equiv 9 \pmod{13}$.

So m is lucky if and only if $2^{F_{m-1}} \in \{3, 9\} \pmod{13}$.

Since 2 is a primitive root modulo 13, and $2^4 = 16 \equiv 3 \pmod{13}$, the above condition is equivalent to requiring $F_{m-1} \in \{4, 8\} \pmod{12}$.

We can verify by inspection (for example, by looking at the terms of the Fibonacci sequence modulo 12, or by looking at the terms of Fibonacci sequence modulo 3 and 4 and using the Chinese Remainder Theorem) that the above condition holds if and only if $m \equiv 7 \pmod{12}$.

Thus m is lucky if and only if it is 7 more than a nonnegative multiple of 12. So the 101^{st} smallest lucky number is

$$100 \cdot 12 + 7 = 1207.$$