Team Round

CHMMC 2016

November 20, 2016

Problem 1. *Let a_n be the nth positive integer such that when n is written in base 3, the sum of the digits of n is divisible by 3. For example, $a_1 = 5$ because $5 = 12_3$. Compute a_{2016}.*

Problem 2. *Consider the 5×5 grid $\mathbb{Z}_5^2 = \{(a, b) : 0 \leq a, b \leq 4\}$. Say that two points $(a, b), (x, y)$ are adjacent if $a - x \equiv -1, 0, 1 \pmod 5$ and $b - y \equiv -1, 0, 1 \pmod 5$. For example, in the diagram, all of the squares marked with $\cdot$ are adjacent to the square marked with $\times$.*

$\times$	$\cdot$			$\cdot$
$\cdot$	$\cdot$			$\cdot$
$\cdot$	$\cdot$			$\cdot$

What is the largest number of $\times$ that can be placed on the grid such that no two are adjacent?

Problem 3. *For a positive integer m, let $f(m)$ be the number of positive integers $q \leq m$ such that $\frac{q^2-4}{m}$ is an integer. How many positive square-free integers $m < 2016$ satisfy $f(m) \geq 16$?*

Problem 4. *Line segments m and n both have length 2 and bisect each other at an angle of $60°$, as shown. A point X is placed at uniform random position along n, and a point Y is placed at a uniform random position along m. Find the probability that the distance between X and Y is less than $\frac{1}{2}$.*

Problem 5. *Given a triangle ABC, let D be a point on segment BC. Construct the circumcircle ω of triangle ABD and point E on ω such that CE is tangent to ω and A, E are on opposite sides of BC(as shown in diagram). If $\angle CAD = \angle ECD$ and $AC = 12, AB = 7$, find AE.*

Problem 6. *For any nonempty set of integers X, define the function*

$$f(X) = \frac{(-1)^{|X|}}{\left(\prod_{k \in X} k\right)^2}$$

where $|X|$ denotes the number of elements in X.

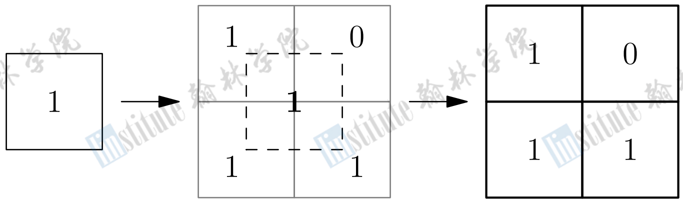
*Consider the set $S = \{2, 3, \ldots, 13\}$. Note that 1 is **not** an element of S.*
Compute

$$\sum_{\substack{T \subseteq S \\ T \neq \emptyset}} f(T).$$

where the sum is taken over all nonempty subsets T of S.

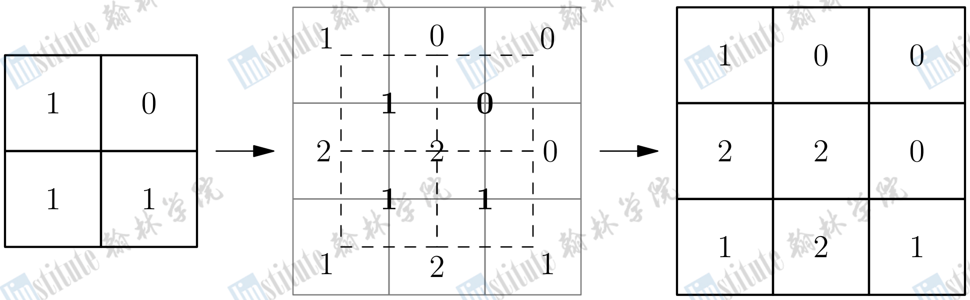
Problem 7. *Consider constructing a tower of tables of numbers as follows. The first table is a one by one array containing the single number* 1.

The second table is a two by two array formed underneath the first table and built as followed. For each entry, we look at the terms in the previous table that are directly up and to the left, up and to the right, and down and to the right of the entry, and we fill that entry with the sum of the numbers occurring there. If there happens to be no term at a particular location, it contributes a value of zero to the sum.



The diagram above shows how we compute the second table from the first.

The diagram below shows how to then compute the third table from the second.



For example, the entry in the middle row and middle column of the third table is equal the sum of the top left entry 1*, the top right entry* 0*, and the bottom right entry* 1 *from the second table, which is just* 2.

Similarly, to compute the bottom rightmost entry in the third table, we look above it to the left and see that the entry in the second table's bottom rightmost entry is 1*. There are no entries from the second table above it and to the right or below it and to the right, so we just take this entry in the third table to be* 1.

We continue constructing the tower by making more tables from the previous tables. Find the entry in the third (from the bottom) row of the third (from the left) column of the tenth table in this resulting tower.

Problem 8. *Let n be a positive integer. If S is a nonempty set of positive integers, then we say S is n-*complete *if all elements of S are divisors of n, and if d_1 and d_2 are any elements of S, then n/d_1 and* $\gcd(d_1, d_2)$ *are in S. How many 2310-complete sets are there?*

Problem 9. *Find the sum of all 3-digit numbers whose digits, when read from left to right, form a strictly increasing sequence. (Numbers with a leading zero, e.g. "087" or "002", are not counted as having 3 digits.)*

Problem 10. *Let ABC be a triangle with circumcircle ω such that $AB = 11$, $AC = 13$, and $\angle A = 30^\circ$. Points D and E are on segments AB and AC respectively such that $AD = 7$ and $AE = 8$. There exists a unique point $F \neq A$ on minor arc AB of ω such that $\angle FDA = \angle FEA$. Compute FA^2.*