# CHMMC 2015 Power Round Problems

In this problem, we will explore the *probabilistic method*, a tool for proving things about deterministic structures by introducing artificial randomness.

## 1 Probability

**Definition 1.1.** *A* counting random variable $X$ *is an object that samples some random process and then returns a positive integer value. We use the symbols* $\Pr(X = n)$ *to denote the probability that* $X$ *will return* $n$ *when sampled.*

**Definition 1.2.** *The expected value of a counting random variable* $X$ *is the average of its outcomes weighted by their probabilities. We denote this by* $\mathbb{E}X$*, and we can define it by the equation*

$$\mathbb{E}X = \sum_{n\in\mathbb{N}} n \cdot \Pr(X = n)$$

*Example* 1.1. The outcome of a 6-sided die roll is a counting random variable $D$. It takes on values between 1 and 6. For each $n$ in the range $1 \leq n \leq 6$, we have $\Pr(D = n) = \frac{1}{6}$. The expected value of $D$ is: $\mathbb{E}D = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6} = \frac{7}{2}$

**Problem 1.1.** *The sum of two independent* 6*-sided dice rolls is a discrete random variable* $D_2$ *with outcomes in the range* $2 \leq n \leq 12$*. What is the expected value of* $D_2$*?*

*Solution* 1.1. The probabilities of each outcome are summarized in this table:

| $n$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\Pr(D = n)$ | $\frac{1}{36}$ | $\frac{2}{36}$ | $\frac{3}{36}$ | $\frac{4}{36}$ | $\frac{5}{36}$ | $\frac{6}{36}$ | $\frac{5}{36}$ | $\frac{4}{36}$ | $\frac{3}{36}$ | $\frac{2}{36}$ | $\frac{1}{36}$ |

Taking the weighted average shows us that the expected value is 7. Notice that this calculation is made easy by the symmetry of the probability table.

**Problem 1.2** (Linearity of expectation)**.** *Let* $X$ *and* $Y$ *be counting random variables. Define* $X + Y$ *to be the result of sampling* $X$*, sampling* $Y$*, and then summing the results. We can find the probability that* $X + Y$ *has a certain value with the following formula:*

$$\Pr(X + Y = n) = \sum_{k\leq n} \Pr(X = k \textit{ and } Y = n - k)$$

*Prove that the expected value of* $X + Y$ *is the sum of the expected values of* $X$ *and* $Y$*.*

*Solution* 1.2. First, we expand out the summation definition.

$$\begin{aligned}
\mathbb{E}(X + Y) &= \sum_{n\in\mathbb{N}} n \cdot \Pr(X + Y = n) \\
&= \sum_{n\in\mathbb{N}} n \cdot \sum_{k\leq n} \Pr(X = k \text{ and } Y = n - k) \\
&= \sum_{n\in\mathbb{N}} \sum_{k\leq n} n \cdot \Pr(X = k \text{ and } Y = n - k)
\end{aligned}$$

Now, we'd like to switch the order of summation, but the index of the inner sum seems to depend on the outer sum. However, this dependence is fictional: if $k \geq n$, then $\Pr(Y = n - k) = 0$. So we can replace the sum over $k \leq n$ with a sum over all $k$.

$$\begin{aligned}
&= \sum_{n\in\mathbb{N}} \sum_{k\in\mathbb{N}} n \cdot \Pr(X = k \text{ and } Y = n - k) \\
&= \sum_{j\in\mathbb{N}} \sum_{k\in\mathbb{N}} (j + k) \cdot \Pr(X = k \text{ and } Y = j)
\end{aligned}$$

We do a change of variable with $j = n - k$ and use linearity to separate out an $X$ component and a $Y$ component:

$$\begin{aligned}
&= \sum_{j\in\mathbb{N}} \sum_{k\in\mathbb{N}} [j \cdot \Pr(X = k \text{ and } Y = j) + k \cdot \Pr(X = k \text{ and } Y = j)] \\
&= \sum_{j\in\mathbb{N}} \sum_{k\in\mathbb{N}} j \cdot \Pr(X = k \text{ and } Y = j) + \sum_{j\in\mathbb{N}} \sum_{k\in\mathbb{N}} k \cdot \Pr(X = k \text{ and } Y = j)
\end{aligned}$$

Now we use the observation that

$$\sum_{j\in\mathbb{N}} \Pr(X = k \text{ and } Y = j) = \Pr(X = k)$$

In other words, the probability that $X$ is observed to be $k$ while $Y$ is observed to be anything else is the same as the probability that $X$ is observed to be $k$.

$$\begin{aligned}
&= \sum_{j\in\mathbb{N}} \sum_{k\in\mathbb{N}} j \cdot \Pr(X = k \text{ and } Y = j) + \sum_{k\in\mathbb{N}} \sum_{j\in\mathbb{N}} k \cdot \Pr(X = k \text{ and } Y = j) \\
&= \sum_{j\in\mathbb{N}} j \cdot \sum_{k\in\mathbb{N}} \Pr(X = k \text{ and } Y = j) + \sum_{k\in\mathbb{N}} k \cdot \sum_{j\in\mathbb{N}} \Pr(X = k \text{ and } Y = j) \\
&= \sum_{j\in\mathbb{N}} j \cdot \Pr(Y = j) + \sum_{k\in\mathbb{N}} k \cdot \Pr(X = k) \\
&= \mathbb{E}X + \mathbb{E}Y
\end{aligned}$$

**Problem 1.3.** *Using linearity of expectation, compute the expected value of the outcome of rolling ten* 6*-sided dice.*

*Solution* 1.3. Let $D_i$ be a counting random variable corresponding to the outcome of the $i$th die roll. Then the outcome of all the dice is given by $T = \sum_{i=1}^{10} D_i$. By linearity of expectation,

$$\mathbb{E}T = \sum_{i=1}^{10} \mathbb{E}D_i = 10\mathbb{E}D = \boxed{35}$$

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## 2 Tournaments without Clear Winners

Suppose $n$ people play a large chess tournament. Each participant plays a match against each other participant, and each match has a winner and a loser. Now suppose we pick $k$ participants and ask them "was there any single participant that won against all $k$ of you?" Is it possible that the answer is always yes, regardless of which $k$ participants we pick? If so, we call the outcome of the tournament *$k$-good*. (By *outcome of the tournament*, we mean the list of outcomes of all of the individual games.)

**Problem 2.1.** *Show that, for any $n > k > 0$, there is an outcome of a tournament with $n$ participants which is not $k$-good.*

*Solution* 2.1. Let one participant win all of their games; call them the winner. Take any set of participants including the winner. There is no participant that won against the winner, so there is no participant that won against all the participants in our set.

**Problem 2.2.** *Suppose $n = 3$ and $k = 1$. Explicitly describe a 1-good outcome of a tournament with 3 participants.*

*Solution* 2.2. Let each participant win exactly one game and lose exactly one game. 1-goodness means exactly that each participant had somebody else win against them, so this outcome is 1-good.

Now, we suppose that we construct a random tournament outcome by deciding each game with a fair coin flip.

**Problem 2.3.** *Pick some set of $k$ people. Let $P_{n,k}$ be the probability that no participant won against all $k$ of them in our random tournament outcome. Calculate $P_{n,k}$.*

*Solution* 2.3. Pick one of the remaining $n-k$ participants, call them $A$. What is the probability that $A$ did not win against all $k$ of the specfied ones? The probability that $A$ won against each one is $\frac{1}{2}$. Each of these victories is independent, so the probability $A$ won against all $k$ of them is $2^{-k}$. The probability that $A$ didn't is $1-2^{-k}$. We need every one of the $n-k$ participants to do this. The probability that each does it is independent from the rest, so this has probability $\boxed{(1-2^{-k})^{n-k}}$.

**Problem 2.4.** *Prove that if $\binom{n}{k}P_{n,k} < 1$, the probability that our random tournament outcome is $k$-good is nonzero. Use the following fact:* $\Pr(A \textit{ or } B) \leq \Pr(A) + \Pr(B)$.

*Solution* 2.4. Our outcome is $k$-good if no set of $k$ people satisfies the property discussed in problem 2.3, which we'll call "badness". However, we cannot use the same technique to calculate this probability, as the events are not independent. (Different sets of $k$ people might have some overlap, and the chance that a set of $k$ participants is bad clearly depends on each of the participants.)

However, we can reframe $k$-goodness as follows: no set of $k$ participants is bad. In symbols:

$$\Pr(\text{outcome is not } k\text{-good}) = \Pr S_1 \text{ is bad or } S_2 \text{ is bad or } \ldots S_{\binom{n}{k}} \text{ is bad}$$

where the $S_i$ are all possible $k$-sized sets of participants. Using the fact of the problem statement, we can bound the probability:

$$\Pr(\text{outcome is not } k\text{-good}) \leq \sum_{i=1}^{\binom{k}{n}} \Pr S_i \text{ is bad} = \binom{k}{n} P_{n,k}$$

If this probability is less than 1, then the probability that the outcome is $k$-good is greater than 0.

**Problem 2.5.** *Conclude that if $\binom{n}{k}P_{n,k} < 1$, then there is a $k$-good outcome for a tournament with $n$ participants. (Hint: In problem 2.4, we gave a coin-flipping construction that yields a $k$-good tournament. Remove the randomness by picking the optimal sequence of coin flips.)*

*Solution* 2.5. We have been reasoning about abstract probabilities, but things are more concrete if we think about them in terms of finite sets. Our tournament is decided by $\binom{n}{2}$ coin flips. Imagine that we iterate through all $2^{\binom{n}{2}}$ possible coin flips and check to see if the resulting tournament is $k$-good. If we count up these $k$-good results and divide by $2^{\binom{n}{2}}$, we'll precisely calculate the probability that a random sequence of coin flips results in a $k$-good tournament. We've already argued that this probability is positive, so the count when we perform this computational experiment must be nonzero.

Now we have a sufficient condition on $n$ and $k$ for there to exist a $k$-good tournament with $n$ participants. Now we ask: are there $k$-good tournaments for every $k$?

**Problem 2.6.** *Prove that, for any fixed $k$, there is sufficiently large $n$ such that a $k$-good tournament with $n$ participants exists. (Hint: Fix $k$, and then prove that $\binom{n}{k}P_{n,k} < 1$ has a solution for $n$.)*

*Solution* 2.6. We want to prove that for fixed $k$, there are sufficently large $n$ such that

$$\binom{n}{k}(1-2^{-k})^{n-k} < 1 \tag{1}$$

A bit of manipulation gives us an equivalent inequality:

$$\begin{aligned}\binom{n}{k}(1-2^{-k})^{n-k} &< 1\\ \binom{n}{k} &< (1-2^{-k})^{k-n}\\ \binom{n}{k} &< \left(\frac{2^k-1}{2^k}\right)^{k-n}\\ \binom{n}{k} &< \left(\frac{2^k}{2^k-1}\right)^{n-k}\\ \binom{n}{k} < n^k &< c^n = \left(\left(\frac{2^k}{2^k-1}\right)^{-k}\right)^n\end{aligned}$$

Where $c > 1$ is a constant. The inequality $\binom{n}{k} < n^k$ can be seen by writing out the fractional expression for $\binom{n}{k}$:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\cdots(n-k+1)}{k!}$$

The numerator has $k$ terms, each of which is at most $n$ and some of which are strictly smaller.

We argue that the exponential $c^n$ is larger than the polynomial $n^k$ for sufficiently large $n$.

Recall that $c > 1$, so that $\log c > 0$. Similarly, assume $n > 1$.

$$n^k < c^n$$
$$k \log n < n \log c$$
$$\frac{k}{\log}c < \frac{n}{\log n}$$

The left hand side is constant while the right hand side is increasing in $n$, so this inequality is satisfied for sufficiently large $n$. Such an $n$ will satisfy inequality 1.

## 3 Covering Dots with Coins

Consider the following game: An adversary places $n$ dots on a piece of paper. You place $n$ coins of equal size on the paper. You win if every dot is covered by a coin.

(More formally: The adversary places $n$ points on the plane. Next, you place $n$ nonoverlapping unit disks on the plane. They may touch on the boundary but not in the interior. You win if every )

**Problem 3.1.** *Prove, using elementary methods, that it is always possible to win if $n = 3$.*

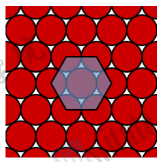
*Solution* 3.1. Suppose that the distance between any pair of the three points is at least 2. Then we can cover the points with three disks, each with its center at one of the points.

Suppose that some pair of points has distance strictly less than 2. Then we cover them with the same disk. If this disk does not also cover the third point, we place another disk to cover it.

We will use the probabilistic method to prove that it is possible to win when $n = 10$.

**Problem 3.2.**

*Suppose we have infinitely many disks and want to cover the whole plane. We do this in the most efficient way we know of: we tile the plane with regular hexagons of side length 2, and put a disk of unit radius at each vertex and at each hexagon's center. What fraction of each hexagon is covered by the disks?*



*Solution* 3.2. The interior angle of a regular hexagon is $\frac{2\pi}{3}$. Therefore, each of the 6 disk sectors contained in a hexagon have area $\frac{1}{3}\pi$. Together with the full disk centered at the center of the hexagon, there is $3\pi$ of area of the hexagon covered by circles. The area of the hexagon is 6 times that of an equilateral triangle of side length 2, which is $\sqrt{3}$. Then the fraction of the hexagon covered by the disks is

$$\frac{3\pi}{6\sqrt{3}} = \boxed{\frac{\pi}{2\sqrt{3}} = \approx 0.907}$$

**Problem 3.3.** *Consider the following random process.*

*There are 10 points in the plane. You lay down an infinite hexagonal tiling of side length 2 with a hexagon centered at the origin. Next, you pick a point $P$ uniformly at random inside the hexagon at the origin. You translate the whole hexagonal grid so that the center of the original hexagon is at point $P$. Finally, you use this hexagonal grid to lay down infinitely many circles as in problem 3.2.*

*Calculate the expected value of the number of points covered by your disks. Justify your answer. (Hint: Make a counting random variable and use linearity of expectation.)*

*Solution* 3.3. Let $X_i$ be a counting random variable which takes on the value 1 if the $i$th point is covered by the disks and 0 if not. Then $X = \sum_{i=1}^{10} X_i$ is a counting random variable whose value is the number of points covered by the disks. $\mathbb{E}X_i$ is just the probability that the $i$th point is covered by the disks. We calculate this probability by a shift of reference frames. Instead of shifting the grid by a random direction in the hexagon, imagine we fix the grid and translate the point. Then the probability that the point lands in a disk is just the fraction of the hexagon that is covered by the disks, or $\frac{\pi}{2\sqrt{3}}$. Then we have:

$$\mathbb{E}X = \mathbb{E}\sum_{i=1}^{10} X_i = 10\mathbb{E}X_1 = \boxed{\frac{5\pi}{\sqrt{3}} \approx 9.07}$$

Now recall the game laid out at the beginning of the problem.

**Problem 3.4.** *Given that the expected value in 3.3 is strictly greater than* 9*, argue that it is always possible to place down just* 10 *disks in order to cover all* 10 *of the adversary's points.*

*Solution* 3.4. Suppose that every way to lay down an infinite hexagonal grid of disks covered at most 9 points. Then the expected value of the number of points covered when laying down the grid randomly would be at most 9. This is false, so there must be at least one way to lay down the infinite grid so as to cover 10 points. When doing so, at most 10 of the disks actually touch any point; just using those, we are done.