CHMMC 2015 Team Problems

November 22, 2015

Problem 0.1. 3 *players take turns drawing lines that connect vertices of a regular* n*-gon. No player may draw a line that intersects another line at a point other than a vertex of the* n*-gon. The last player able to draw a line wins. For how many* n *in the range* $4 \leq n \leq 100$ *does the first player have a winning strategy?*

Problem 0.2. *You have 4 game pieces, and you play a game against an intelligent opponent who has 6. The rules go as follows: you distribute your pieces among two points* a *and* b*, and your opponent simultaneously does as well (so neither player sees what the other is doing). You win the round if you have more pieces than them on either* a *or* b*, and you lose the round if you only draw or have fewer pieces on both. You play the optimal strategy, assuming your opponent will play with the strategy that beats your strategy most frequently. What proportion of the time will you win?*

Problem 0.3. *A trio of lousy salespeople charge increasing prices on tomatoes as you buy more. The first charges you* x_1 *dollars for the* x_1*th tomato you buy from him, the second charges* x_2^2 *dollars for the* x_2*th tomato, and the third charges* x_3^3 *dollars for the* x_3*th tomato. If you want to buy* 100 *tomatoes for as cheap as possible, how many should you buy from the first salesperson?*

Problem 0.4. *Let* $P(x) = x^{16} - x^{15} + \cdots - x + 1$*, and let* p *be a prime such that* $p - 1$ *is divisible by* 34 *(*$p = 103$ *is an example). How many integers* a *between* 1 *and* $p - 1$ *inclusive satisfy the property that* $P(a)$ *is divisible by* p*?*

Problem 0.5. *Felix is playing a card-flipping game.* n *face-down cards are randomly colored, each with equal probability of being black or red. Felix starts at the* 1*st card. When Felix is at the* k*th card, he guesses its color and then flips it over. For* $k < n$*, if he guesses correctly, he moves onto the* $(k+1)$*-th card. If he guesses incorrectly, he gains* k *penalty points, the cards are replaced with newly randomized face-down cards, and he moves back to card* 1 *to continue guessing. If Felix guesses the* n^{th} *card correctly, the game ends.*

What is the expected number of penalty points Felix earns by the end of the game?

Problem 0.6. *The icosahedron is a convex, regular polyhedron consisting of* 20 *equilateral triangle for faces. A particular icosahedron given to you has labels on each of its vertices, edges, and faces. Each minute, you uniformly at random pick one of the labels on the icosahedron. If the label is on a vertex, you remove it. If the label is on an edge, you delete the label on the edge along with any labels still on the vertices of that edge. If the label is on a face, you delete the label on the face along with any labels on the edges and vertices which make up that face. What is the expected number of minutes that pass before you have removed all labels from the icosahedron?*

Problem 0.7. *Let I be the incenter and let Γ be the incircle of $\triangle ABC$, and let $P = \Gamma \cap BC$. Let Q denote the intersection of Γ and the line passing through P parallel to AI. Let ℓ be the tangent line to Γ at Q and let $\ell \cap AB = S, \ell \cap AC = R$. If $AB = 7, BC = 6, AC = 5$, what is RS?*

Problem 0.8.

$$\text{Let } f(n) = \sum_{d=1}^{n} \left\lfloor \frac{n}{d} \right\rfloor \text{ and } g(n) = f(n) - f(n-1).$$

For how many n from 1 to 100 inclusive is $g(n)$ even?

Problem 0.9. *Let T be a 2015×2015 array containing the integers $1, 2, 3, \ldots, 2015^2$ satisfying the property that $T_{i,a} > T_{i,b}$ for all $a > b$ and $T_{c,j} > T_{d,j}$ for all $c > d$ where $1 \le a, b, c, d \le 2015$ and $T_{i,j}$ represents the entry in the ith row and jth column of T. How many possible values are there for the entry at $T_{5,5}$?*

Problem 0.10. *Let $\mathcal{P}$ be the parabola in the plane determined by the equation $y = x^2$. Suppose a circle $\mathcal{C}$ in the plane intersects $\mathcal{P}$ at four distinct points. If three of these points are $(-28, 784), (-2, 4)$, and $(13, 169)$, find the sum of the distances from the focus of $\mathcal{P}$ to all four of the intersection points.*