CHMMC 2015 Power Round Problems

In this problem, we will explore the *probabilistic method*, a tool for proving things about deterministic structures by introducing artificial randomness.

1 Probability

Definition 1.1. *A* counting random variable X *is an object that samples some random process and then returns a positive integer value. We use the symbols* $\Pr(X = n)$ *to denote the probability that* X *will return* n *when sampled.*

Definition 1.2. *The expected value of a counting random variable* X *is the average of its outcomes weighted by their probabilities. We denote this by* $\mathbb{E}X$*, and we can define it by the equation*

$$\mathbb{E}X = \sum_{n \in \mathbb{N}} n \cdot \Pr(X = n)$$

Example 1.1. The outcome of a 6-sided die roll is a counting random variable D. It takes on values between 1 and 6. For each n in the range $1 \leq n \leq 6$, we have $\Pr(D = n) = \frac{1}{6}$. The expected value of D is: $\mathbb{E}D = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6} = \frac{7}{2}$

Problem 1.1. *The sum of two independent* 6*-sided dice rolls is a discrete random variable* D_2 *with outcomes in the range* $2 \leq n \leq 12$*. What is the expected value of* D_2*?*

Problem 1.2 (Linearity of expectation)**.** *Let* X *and* Y *be counting random variables. Define* $X + Y$ *to be the result of sampling* X*, sampling* Y*, and then summing the results. We can find the probability that* $X + Y$ *has a certain value with the following formula:*

$$\Pr(X + Y = n) = \sum_{k \leq n} \Pr(X = k \textit{ and } Y = n - k)$$

Prove that the expected value of $X + Y$ *is the sum of the expected values of* X *and* Y*.*

Problem 1.3. *Using linearity of expectation, compute the expected value of the outcome of rolling ten* 6*-sided dice.*

2 Tournaments without Clear Winners

Suppose n people play a large chess tournament. Each participant plays a match against each other participant, and each match has a winner and a loser. Now suppose we pick k participants and ask them "was there any single participant that won against all k of you?" Is it possible that the answer is always yes, regardless of which k participants we pick? If so, we call the outcome of the tournament *k-good*. (By *outcome of the tournament*, we mean the list of outcomes of all of the individual games.)

Problem 2.1. *Show that, for any* $n > k > 0$*, there is an outcome of a tournament with* n *participants which is not* k*-good.*

Problem 2.2. *Suppose $n = 3$ and $k = 1$. Explicitly describe a 1-good outcome of a tournament with 3 participants.*

Now, we suppose that we construct a random tournament outcome by deciding each game with a fair coin flip.

Problem 2.3. *Pick some set of k people. Let $P_{n,k}$ be the probability that no participant won against all k of them in our random tournament outcome. Calculate $P_{n,k}$.*

Problem 2.4. *Prove that if $\binom{n}{k}P_{n,k} < 1$, the probability that our random tournament outcome is k-good is nonzero. Use the following fact:* $\Pr(A \text{ or } B) \leq \Pr(A) + \Pr(B)$.

Problem 2.5. *Conclude that if $\binom{n}{k}P_{n,k} < 1$, then there is a k-good outcome for a tournament with n participants. (Hint: In problem 2.4, we gave a coin-flipping construction that yields a k-good tournament. Remove the randomness by picking the optimal sequence of coin flips.)*

Now we have a sufficient condition on n and k for there to exist a k-good tournament with n participants. Now we ask: are there k-good tournaments for every k?

Problem 2.6. *Prove that, for any fixed k, there is sufficiently large n such that a k-good tournament with n participants exists. (Hint: Fix k, and then prove that $\binom{n}{k}P_{n,k} < 1$ has a solution for n.)*

3 Covering Dots with Coins

Consider the following game: An adversary places n dots on a piece of paper. You place n coins of equal size on the paper. You win if every dot is covered by a coin.

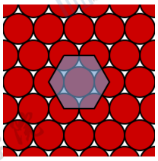
(More formally: The adversary places n points on the plane. Next, you place n nonoverlapping unit disks on the plane. They may touch on the boundary but not in the interior. You win if every)

Problem 3.1. *Prove, using elementary methods, that it is always possible to win if $n = 3$.*

We will use the probabilistic method to prove that it is possible to win when $n = 10$.

Problem 3.2.

Suppose we have infinitely many disks and want to cover the whole plane. We do this in the most efficient way we know of: we tile the plane with regular hexagons of side length 2, and put a disk of unit radius at each vertex and at each hexagon's center. What fraction of each hexagon is covered by the disks?



Problem 3.3. *Consider the following random process.*

There are 10 points in the plane. You lay down an infinite hexagonal tiling of side length 2 with a hexagon centered at the origin. Next, you pick a point P uniformly at random inside the hexagon at the origin. You translate the whole hexagonal grid so that the center of the original hexagon is at point P. Finally, you use this hexagonal grid to lay down infinitely many circles as in problem 3.2.

Calculate the expected value of the number of points covered by your disks. Justify your answer. (Hint: Make a counting random variable and use linearity of expectation.)

Now recall the game laid out at the beginning of the problem.

Problem 3.4. *Given that the expected value in 3.3 is strictly greater than 9, argue that it is always possible to place down just 10 disks in order to cover all 10 of the adversary's points.*