Tiebreaker Round Solutions

2014 CHMMC

Solution 1.

$$\boxed{\frac{65}{72}}$$

Solution 2. We have $y = b(a+d)$ and $z = c(a+d)$, so $b = y/(a+d)$ and $c = z/(a+d)$. Thus b and c are entirely determined by a and d. Plugging these in gives $a^2 + bc = a^2 + yz(a+d)^2$ and $d^2 + bc = d^2 + yz(a+d)^2$. The equations $a^2 + yz(a+d)^2 = x$ and $d^2 + yz(a+d)^2 = w$ clearly have two roots with respect to a and d since $x, w \neq 0$. Therefore given a valid a, there are two possible values of d, and given a valid d there are two possible values of a. This gives a total of $\boxed{4}$ possibilities. Alternatively, use diagonalization; by solving the roots of the characteristic polynomial we get $\lambda = 3 \pm \sqrt{7}$:

$$\begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix} = P \begin{bmatrix} 3+\sqrt{7} & 0 \\ 0 & 3-\sqrt{7} \end{bmatrix} P^{-1}$$

which must give the roots

$$\begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix} = P \begin{bmatrix} \pm\sqrt{3+\sqrt{7}} & 0 \\ 0 & \pm\sqrt{3-\sqrt{7}} \end{bmatrix} P^{-1}$$

where the $\pm$ are independent. This again gives 4 possibilities.

Solution 3. If you're not in one of these positions, you can move to one of them: $1, 3, 6$ removes 1 bean, 4 removes 4 beans and 7 removes 7 beans.

If you are in one of these positions, you can't move to one of them: $0 \mapsto \{6,4,1\}; 2 \mapsto \{1,6,3\}; 5 \mapsto \{4,1,6\}$

The winning strategy for this game is to always move to $n \equiv 0, 2, 5 \pmod 8$ if possible, so player 2 wins whenever $n \equiv 0, 2, 5 \pmod 8$. There are $\boxed{14}$ numbers between 2014 and 2050 satisfying this.

Solution 4. $f(n,0,0) = n^2$.

The second coordinate keeps a running total of the sum of the various values of the third coordinate as the first coordinate goes to 0. In other words, $f(n,0,0) = \sum_{i=1}^{n}(2i-1) = \boxed{n^2}$

Solution 5. Let the expression be x. Take the natural log to give

$$\ln x = \sum_{n=1}^{\infty} \frac{n}{3^n} \ln 3$$

and the sum reduces to

$$\ln x = \frac{1}{3} \ln 3 (3/2)^2$$

$$x = \boxed{3^{3/4}}$$