Team Round Solutions

2014 CHMMC

Solution 1. Call a hexagon (hex) a "centerpiece" of a chunk if it is in one of the two chunk's hexes adjacent to all the other hexes in the chunk. If we sum over all hexes the number of chunks that the hex is a centerpiece of, we will clearly get twice the number of chunks. Each corner is a centerpiece of only one, each edge is a centerpiece of two, and each middle piece is a centerpiece of six. Since there are 19 middle pieces, 12 edge pieces, and 6 corner pieces, this gives $144/2 = \boxed{72}$ chunks.

Solution 2. The volume of one regular tetrahedron is $\frac{s^3}{6\sqrt{2}}$ where s is side length; this can be computed by $V = \frac{1}{3}Bh$ and finding the base and height of a tetrahedron. Since the tetrahedrons are regular, any edge's nearest point to the center is at its midpoint. Since the overlapping tetrahedrons have the same side length, (you have to visualize the setup here) we see that any edge of the first tetrahedron is closest to the center at the same point as an edge of the second tetrahedron. This means that the edges intersect, so the resulting shape is just a tetrahedron with a smaller tetrahedrons coming out of each of the four faces. Each of the smaller tetrahedrons is similar to the original (since it is the original sliced by a plane parallel to the base), and clearly has half the side length of the original. Therefore the volume of this union is $1 + 4(\frac{1}{2})^3 = 3/2$ as much as that of one tetrahedron, and plugging in $s = 2$ gives $2/\sqrt{2} = \boxed{\sqrt{2}}$.

Solution 3. We see that each friendship is shown in all but two of these graphs, so the total number of edges (friendships) is $\frac{1}{4}$ times the sum of edges in all graphs. This indicates there are 11 friendships within the group. The number of friendships Sue has is the number missing in the first graph; this gives $11 - 7 = \boxed{4}$. The problem can also be done by deducing which node Sue is in various graphs.

Solution 4. Let $a_n = \prod_{i=1}^{n} b_i$. Then we have $a_1 = 1, a_{n+1} = a_n + \frac{1}{n} - \frac{1}{n+1}$. Telescoping, we see that most of the $1/n - 1/(n+1)$ terms cancel, leaving $a_n = 2 - 1/n$, or

$$b_n = \frac{2 - 1/n}{2 - 1/(n-1)}$$

This gives $b_{12} = \boxed{253/252}$.

Solution 5. On each question the average student must have answered a question in common with $(15)/5 - 1 = 2$ other students. Since each student has only 28 answers in common with anyone else total, this can only go on for $\boxed{14}$ questions. We see that such an arrangement is possible by having each answer receive an equal number of students each time and having the groups of students who answer together shuffle after every other question.

Solution 6. The easiest way is to work with complex numbers, but the same can be done in (x, y) coordinates. The transformation is just $Tz \to 1/z$, so for instance the line $x = 1$ is $z = 1 + yi$, which maps to $\frac{1-yi}{1+y^2}$. If we translate by $1/2$ to the left we get $\frac{1}{2}\frac{1-2yi-y^2}{1+y^2}$. This has constant norm $1/2$, so a line of distance 1 from the origin must map to a segment of a circle of radius $1/2$. Trying points gives $T(1,1) \to (1/2, -1/2), T(1,0) \to (1,0), T(1,-1) \to (1/2, 1/2)$. Thus the right side of the square maps to a semicircle of radius $1/2$, as do the other sides. It is easy to see that the resulting shape is a square of side length one with four semicircles of radius $1/2$, giving an area of $\boxed{1 + \pi/2}$.

Solution 7. Observe that

$$P(x)Q(x) = \prod_{k=0}^{n} (x^{2\cdot 3^k} + x^{-2\cdot 3^k} + 1)$$

Consider the coefficient of x^k upon expanding. Observe that if k is odd, then since every power in the product is even, the coefficient must be zero. On the other hand, if k is even, then the coefficient will be 1 since there is exactly one way to write the integer $\frac{k}{2}$ in base 3. Since there are $\frac{3^n+1}{2}$ even numbers in the range $[0, 3^n]$, it follows that the answer is

$$\boxed{\frac{3^n + 1}{2}}$$

Solution 8. Given that one side of a triangle has length 2, the only possible lengths of the other two are $\{3,3\}$, $\{3,4\}$, $\{4,5\}$, or $\{5,5\}$.

If two triangle sides in a pyramid share the edge of length 2, we can only combine $\{3,3\}$ and $\{5,5\}$ (call this Case 1), $\{3,3\}$ and $\{4,5\}$ (Case 2), or $\{3,4\}$ and $\{5,5\}$ (Case 3).

Case 1 yields a unique pyramid with 2 right angles. The volume is $\frac{8\sqrt{2}}{3}$.

Case 2 yields two possible pyramids, but it is easy to see that they share the same volume. It is also easy to see that its volume is not as great as it is in Case 1.

Case 3 yields two possible pyramids, also of equal volume. With a slight bit of trouble one can see that it is bounded above by a volume lower than it is on Case 1.

Therefore the greatest volume is $\boxed{\frac{8\sqrt{2}}{3}}$

Solution 9. The first thing to notice is that for $n > 2$, n repeats at least twice due to the $\binom{n}{1}$ and $\binom{n}{n-1}$ diagonals. Also, the triangle is symmetric. Therefore the only way to occur exactly 3 times is to occur twice on those diagonals and once in the middle column, $\binom{2k}{k}$. We now have to calculate how many such $\binom{2k}{k}$ there are under 100,000. The first terms (excluding $\binom{2}{1}$ which only occurs once), go 6, 20, 70, 252. We notice that the ratio between them is slightly less than 4, which makes sense since we are multiplying by about $(2k)^2/(k^2) = 4$ each time. Also the ratio will increase towards 4 as k increases, so for $k > 5$, it is at least $252/70 = 3.6$. Extrapolating gives the next values of $\binom{2k}{k}$ to be roughly 1000, 4000, 16000, 64000, 256000. Even if the ratio remains as low as 3.6 (which it doesn't), this last term is still at least $(9/10)^5 * 256000 > 128000$. We see that the last term which could repeat exactly 3 times and be under 100,000 is $\binom{18}{9} \approx 64000$. This gives up to 8 values which could repeat 3 times. Using the assumption, the answer must be $\boxed{8}$.

Solution 10. Notice the following invariant: moving via a path preserves whether the product of the indices is a square or not. By this we mean that if $m_1 n_1$ is a square and (m_1, n_1) has a path to (m_2, n_2), then $m_2 n_2$ must be a square according to either rule. Since $(1,1)$ has the product of its components $1 * 1 = 1$ as a perfect square, the number of (m,n) connected to $(1,1)$ is just the number of lattice points whose product of indices is a square. Consider all such lattice points whose indices have a greatest common divisor of 1. These are simply the set of ordered pairs of squares under 125. Sort these by largest element:

$$(1,1)$$

$$(4,1),(1,4)$$

$$(9,1),(9,4),(4,9),(1,9)$$

$$\vdots$$

We see that for $n > 1$, there are $2\varphi(n)$ of these for largest square n^2 where $\varphi(n)$ is Euler's totient function. For $n = 1$ there is one. Each of these will be repeated $\lfloor 125/n^2 \rfloor$ times for the multiples (kn^2, km^2). With some thinking, we can be certain that this counting method includes all such coordinates. This gives $125 + 2\lfloor 125/4 \rfloor + 4\lfloor 125/9 \rfloor + 4\lfloor 125/16 \rfloor + \ldots 12\lfloor 125/196 \rfloor$ pairs connected to $(1,1)$. Summing gives $\boxed{391}$.