Team Round

Caltech Harvey Mudd Mathematics Competition

November 23, 2013

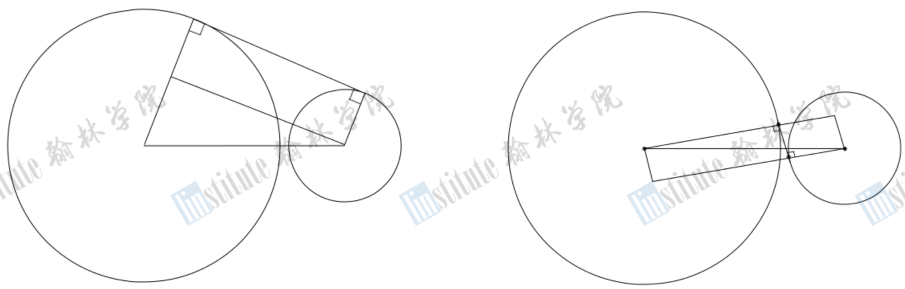
1. In how many ways can you rearrange the letters of 'Alejandro' such that it contains one of the words 'ned' or 'den'?

   Solution: The answer is 5040.

   Consider (ned) as one character. There are $7!/2$ ways to permute aljaro(ned). Multiply by 2 to count the (den) solutions. Hence, the answer is 5040.

2. Two circles of radii 7 and 17 have a distance of 25 between their centers. What is the difference between the lengths of their common internal and external tangents (positive difference)?

   Solution: The answer is $5\sqrt{21} - 7$.



   As seen in the left figure, the external tangent is $\sqrt{25^2 - (17-7)^2} = 5\sqrt{21}$, and as seen in the right figure, the length of the common internal tangent is $\sqrt{25^2 - (7+17)^2} = 7$. So we have $5\sqrt{21} - 7$ as our final answer.

3. Let p_n be the product of the nth roots of 1. For integral $x > 4$, let $f(x) = p_1 - p_2 + p_3 - p_4 + \cdots + (-1)^{x+1}p_x$. What is $f(2010)$?

   Solution: The answer is 2010.

   The nth roots of unity satisfy the polynomial equation $x^n - 1 = 0$. By Vieta's formula, the product of the roots is given by $(-1)^n \frac{-1}{1} = (-1)^{n+1}$. So the product of the roots of unity is 1 when n is odd and -1 when n is even. Thus $f(x) = x$ and $f(2010) = 2010$.

4. The numbers 25 and 76 have the property that when squared in base 10, their squares also end in the same two digits. A positive integer that has at most 3 digits when expressed in base 21 and also has the property that its base 21 square ends in the same 3 digits is called amazing. Find the sum of all amazing numbers. Express your answer in base 21.

   Solution: The answer is 1002.

   It suffices to solve the congruence $a(a-1) \equiv 0 \pmod{21^3}$ for $0 < a < 21^3$. By the Chinese Remainder Theorem, this is equivalent to $a(a-1) \equiv 0 \pmod{7^3}$ and $a(a-1) \equiv 0 \pmod{3^3}$. Since $a, a-1$ are relatively prime, we have four cases.

   Case 1: $21^3 \mid a$. Not possible.

   Case 2: $21^3 \mid a - 1$. Only $a = 1$ works.

   Case 3: $7^3 \mid a$, $3^3 \mid a - 1$. Note that $7^3, 2 \cdot 7^3, \ldots, (3^3 - 1)7^3$ form a complete set of nonzero residues mod 3^3, so exactly one of them is a solution.

   Case 4: $7^3 \mid a - 1$, $3^3 \mid a$. Again, $3^3, 2 \cdot 3^3, \ldots, (7^3 - 1)3^3$ form a complete set of nonzero residues mod 7^3, so exactly one of them is a solution.

There are three amazing numbers total. The problem can be finished by noting that if a is a solution, then $21^3 + 1 - a$ is also a solution. The two answers in cases 3 and 4 must be paired like this and have sum $21^3 + 1$, while the solution $a = 1$ is paired to 21^3 which is too high. So the sum is $1 + (21^3 + 1) = 21^3 + 2$, which is 1002 in base 21.

5. Compute the number of lattice points bounded by the quadrilateral formed by the points $(0, 0)$, $(0, 140)$, $(140, 0)$, and $(100, 100)$ (including the quadrilateral itself). A lattice point on the x-y plane is a point (x, y), where both x and y are integers.

   Solution: The answer is 14580.

   First, note that the region can be divided into three partitions: the square S enclosed by the points $(0, 0)$, $(0, 100)$, $(100, 100)$, and $(100, 0)$; the triangle T_1 enclosed by the points $(100, 0)$, $(100, 100)$, and $(140, 0)$; and the triangle T_2 enclosed by the points $(0, 140)$, $(0, 100)$, and $(100, 100)$. We will find the number of lattice points in each of these partitions, then combine to get the total answer. For further terminology, we will use $I(X)$ to be the number of lattice points strictly in the interior of the shape X, and $B(X)$ to be the number of lattice points strictly in the boundary of X, and $A(X)$ to be the total number of lattice points in X.

   We will find $A(T_1)$, equal to $A(T_2)$, and then $A(S)$.

   Note that $2 \cdot I(T_1) + B(\text{triangle's slant}) = I(\text{rectangle formed by a triangle and its mirror image})$.

   Also note $B(\text{slant line}) = \lfloor (|m_{slant}|(\Delta x) \rfloor - 1$.

   Solving for $I(T_1)$, we get:

$$(I(T_1) = \frac{I(\text{rectangle}) - B(\text{slant})}{2}.$$

   Note that the number of lattice points strictly within a rectangle of dimensions (x, y) is $(x - 1)(y - 1)$; substitute and evaluate into the equation above, using $x = 100$ and $y = 40$.

$$I(T_1) = \frac{(101)(41) + (\frac{2}{5} \cdot 100 - 1)}{2} = 2090 \text{ points.}$$

   Since the two triangles are identical by symmetry, we multiply this figure by two to get 4180 points strictly within both triangles.

   When we add the points in the square to this figure $(100 - 1)(100 - 1)$, we must add the points on the triangle-square interfaces but not the corners.

$$B(\text{interface points}) = 101 \text{ points per edge} \cdot 2 \text{ edges} - 3 \text{ corner points} = 199 \text{ points.}$$

   Summing up the square points, the interface points, and the triangle points, we get a total of $(10201 + 199 + 4180) = 14580$ points.

6. Let $a_1 < a_2 < a_3 < \ldots < a_n < \ldots$ be positive integers such that, for $n = 1, 2, 3, \ldots$,

$$a_{2n} = a_n + n.$$

   Given that if a_n is prime, then n is also, find a_{2014}.

   Solution: The answer is 2014.

   Using the assumption that $a_{2n} = a_n + n$, we check $n = 1$ and $n = 2$. Noticing that $a_2 = a_1 + 1$, and that $a_4 = a_2 + 2$, I would like to prove the induction that $a_n = a_1 + n - 1$.

Then, $a_1 + 1 = a_2 < a_3 < a_1 + 3 = a_4$. Since each a is an integer, it follows that $a_3 = a_2 + 2$. Now we move to attempting to prove the induction.

$a_k = a_1 + k - 1$
$a_{2k} = a_1 + 2k - 1$ (since $a_{2n} = a_n + n$), but:
$a_1 + k - 1 = a_k < a_{k+1} < a_{k+2} < \ldots < a_{2k-1} < a_{2k} = a_1 + 2k - 1$.
Since these are integers, it follows that k is increasing by 1 for each increasing a. Thus:
$a_{k+1} = a_k + 1 = a_1 + k$.
This completes the induction. We have shown that for all numbers $n \geq 1, a_n = a_1 + n - 1$.

Now we have to find a_1.

Since a_1 has to be a positive integer, it must be greater than or equal to 1. Let us assume that it is greater than 1. That would mean that

$(a_1 + 1)! + 2, (a_1 + 1)! + 3, \ldots, (a_1 + 1)! + a_1 + 1$
are all composite values. Let p be the smallest prime number greater than $(a_1 + 1)! + a_1 + 1$. Let $n = p - a_1 + 1$. So $p = a_1 + n - 1 = a_n$. Given the initial assumption that if a_n is prime, n is also, we have that n is prime. But you notice that:
$(a_1 + 1)! + 2 < p - a_1 + 1$ (because $p > (a_1 + 1)! + a_1 + 1$) and that

$p - a_1 + 1 \leq p - 1$, (equal if and only if $a_1 = 2$)

Since p is the smallest prime number greater than the numbers we proved to be composite, and all numbers with which we are dealing are integers, $p - 1$ must be composite, thus $p - a_1 + 1$ must also be composite. Recall that this equals n. We have shown that n is prime and n is composite, which is a contradiction. So we know that $a_1 = 1$. Now:

$a_n = a_1 + n - 1 = n$. Thus $a_{2014} = 2014$.

7. The points $(0, 0)$, $(a, 5)$, and $(b, 11)$ are the vertices of an equilateral triangle. Find ab.
Solution: The answer is $\frac{17}{3}$.

Looking in the complex plane, we see that $b + 11i$ is the 60 degree rotation of $a + 5i$. Thus:

$(a + 5i)(\text{cis } 60) = b + 11i$

Equating real and imaginary parts and solving the resulting system gives $a = \pm\frac{17}{\sqrt{3}}$ and $b = \pm\frac{1}{\sqrt{3}}$. Multiplying them results in $ab = \frac{17}{3}$.

8. Two kids A and B play a game as follows: from a box containing n marbles ($n > 1$), they alternately take some marbles for themselves, such that:

   1. A goes first.
   2. The number of marbles taken by A in his first turn, denoted by k, must be between 1 and $n - 1$, inclusive.
   3. The number of marbles taken in a turn by any player must be between 1 and k, inclusive.

   The winner is the one who takes the last marble. Determine all natural numbers n for which A has a winning strategy.

   Solution: The answer is all n where $n + 1$ is not prime.

   The key observation is that either player can guarantee that the number of marbles taken in his turn and the previous turn (which is his opponent's) add up to $k + 1$. Thus, the winner is decided by whether $k+1$ divides $n-k$: if it does, A can guarantee that the number of marbles left after his turns is always divisible by $k+1$, and A will be the winner. Otherwise, in his first turn, B can take $r = (n-k) \mod (k+1)$ marbles. Then, he is the one who can guarantee that the number of marbles left after his turns is always divisible by $k+1$.

So the question is whether A can choose k such that $k+1$ divides $n-k$. Clearly, he can do so if and only if $n+1$ is not a prime.

9. A 7×7 grid of unit-length squares is given. Twenty-four 1×2 dominoes are placed in the grid, each covering two whole squares and in total leaving one empty space. It is allowed to take a domino adjacent to the empty square and slide it lengthwise to fill the whole square, leaving a new one empty and resulting in a different configuration of dominoes. Given an initial configuration of dominoes for which the maximum possible number of distinct configurations can be reached through any number of slides, compute the maximum number of distinct configurations.

Solution: The answer is 16.

Coordinatize the board with corners $(0,0),(0,6),(6,6),(6,0)$. Each move causes the empty square at (x,y) to move two units in some direction to $(x \pm 2, y)$ or $(x, y \pm 2)$. So the maximum possible number of different positions for the empty squares is 16, in the case that the empty square initially had even coordinates. We now show that it is impossible for two configurations with the same empty space to be reachable from each other through sliding. This implies that the maximum number of distinct configurations is at most 16.

Suppose two configurations, with the same empty square, have some sequence of slides $m_1, m_2, \ldots m_k$ taking the first configuration to the second, and take the pair such that k is minimized. Let the empty square be at (x,y), and WLOG m_1 moves the empty square $(x+2,y)$ (rotate the board if not). That means after m_1 there is a domino on (x,y), $(x+1,y)$. The final slide must uncover (x,y), meaning this domino must move again on the final move (if it moves earlier, minimality of k is contradicted). At any given moment this domino can slide only if either $(x-1,y)$ or $(x+2,y)$ is empty. $(x-1,y)$ is never empty by parity, so $(x+2,y)$ must be empty. Then the configuration after slide m_1 leaves $(x+2,y)$ empty, and the configuration after moves $m_1, m_2, \ldots m_{k-1}$ also leaves $(x+2,y)$ empty and is different. So $m_2, \ldots m_{k-1}$ are a sequence of moves between two configurations with the same empty square, contradicting minimality of k.

Finally, consider a board in which $(0,0)$ is empty, all spaces with x-coordinate 0 are filled by vertical dominoes, and the rest of the board is tiled with horizontal ones. It is clear 16 configurations are reachable from here.

10. Compute the lowest positive integer k such that none of the numbers in the sequence $\{1, 1+k, 1+k+k^2, 1+k+k^2+k^3, \cdots\}$ are prime.

Solution: The answer is 9.

Notice that for odd powers n, $1+k+k^2+\cdots+k^n$ is divisible by $1+k$. Even powers $2m$ will factor if k is a square; specifically, if $k=j^2$, then $1+k+k^2+\cdots+k^{2m} = 1+j^2+j^4+\cdots+j^{4m}$ $= (1+j+j^2+\cdots+j^{2m})(1-j+j^2-\cdots+j^{2m})$. We need $1+k$ to be a composite number. Thus 1 and 4 will not work and k must be at least 9, so $k=9$. Note: Remarkably, we can show that there exists a prime within the first 5 terms of the sequence for every k less than 9! For $k=1$, $2=1+1$ is prime. For $k=2$, $3=1+2$ is prime. For $k=3$, $13=1+3+3^2$ is prime. For $k=4$, $5=1+4$ is prime. For $k=5$, $31=1+5+5^2$ is prime. For $k=6$, $43=1+6+6^2$ is prime. For $k=7$, $2801=1+7+7^2+7^3+7^4$ is prime. For $k=8$, $73=1+8+8^2$ is prime. Thus $k=9$ is the smallest value that could possibly work (and it does!).