Caltech Harvey Mudd Mathematics Competition

Tiebreaker Round

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November 23, 2013

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1. In the diagram below, point A lies on the circle centered at O. AB is tangent to circle O with $\overline{AB} = 6$. Point C is $\frac{2\pi}{3}$ radians away from point A on the circle, with BC intersecting circle O at point D. The length of BD is 3. Compute the radius of the circle.

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Solution: The answer is $-\sqrt{3} + \sqrt{39}$.

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First, using the tangent-secant power theorem, find \overline{BC} :

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$$\overline{BC} = \frac{\overline{AB}^2}{\overline{BD}} = \frac{6^2}{3} = 12.$$

Then, knowing that $\angle AOC$ is $\frac{2\pi}{3}$, that $\angle OAB = \frac{\pi}{2}$ (tangent radius relationship), and that $\overline{AO} = \overline{CO}$ (both radii), it can be determined that $\angle BAC = \frac{2\pi}{3}$:

$$\angle OAC = \pi - \angle AOC - \angle OCA = \angle OCA = \frac{\pi - \frac{2\pi}{3}}{2} = \frac{\pi}{6},$$
$$\angle BAC = \angle OAB + \angle OAC = \frac{\pi}{2} + \frac{\pi}{6} = \frac{2\pi}{3}.$$

 $\angle OAC = \pi - \angle AOC$ $\angle BAC = \angle OAB + (\Box AAB + \Box OAB + \Box OAB + (\Box AAB + \Box OAB + \Box OAB + (\Box AAB + \Box OAB + \Box OAB + (\Box AAB + \Box AAB + (\Box AAB + (\Box AAB + \Box AAB + (\Box AAB + \Box AAB + (\Box AAB + \Box AAB + (\Box AAB + (\Box AAB + \Box AAB + (\Box AAB + (\Box AAB + \Box AAB + (\Box AAB + (\Box AAB + \Box AAB + (\Box AAB$

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$$\overline{BC}^{2} = \overline{AB}^{2} + \overline{AC}^{2} - 2\overline{AB} \ \overline{AC} \cos(\angle BAC)$$

$$144 = 36 + \overline{AC}^{2} + 6\overline{AC}$$

$$\overline{AC} = \frac{-6 + \sqrt{6^{2} + 4 \times 108}}{2}$$

$$\overline{AC} = \frac{-6 + 2\sqrt{117}}{2}$$

$$\overline{AC} = -3 + 3\sqrt{13}$$

Finally, bisect \overline{AC} and use a 30-60-90 triangle to find \overline{OA} , the radius:

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$$\overline{OA} = \frac{2}{\sqrt{3}} \cdot \frac{\overline{AC}}{2} = -\sqrt{3} + \sqrt{39}.$$
 Hence, the answer is $-\sqrt{3} + \sqrt{39}$.

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2. Suppose the roots of

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$$x^4 - 3x^2 + 6x - 12 = 1$$

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 $x^{z} =$ are α, β, γ , and δ . What is the value of

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$$x^{4} - 3x^{2} + 6x - 12 = 1$$

nat is the value of
$$\frac{\alpha + \beta + \gamma}{\delta^{2}} + \frac{\alpha + \delta + \gamma}{\beta^{2}} + \frac{\alpha + \beta + \delta}{\gamma^{2}} + \frac{\delta + \beta + \gamma}{\alpha^{2}} ?$$

Solution: The answer is $-\frac{1}{2}$.

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Because α , β , γ , and δ are the roots of the above equation, we know that their sum is the negative of the coefficient of the x^3 term, which is 0. Hence, we can simplify:

 $\frac{\alpha + \beta + \gamma}{\delta^2} + \frac{\alpha + \delta + \gamma}{\beta^2} + \frac{\alpha + \beta + \delta}{\gamma^2} + \frac{\delta + \beta + \gamma}{\alpha^2} = \frac{0 - \delta}{\delta^2} + \frac{0 - \beta}{\beta^2} + \frac{0 - \gamma}{\gamma^2} + \frac{0 - \alpha}{\alpha^2}$ $= -\frac{1}{\delta} - \frac{1}{\beta} - \frac{1}{\gamma} - \frac{1}{\alpha}$ 面影曲新林塔梯 $=-\frac{-6}{12}=-\frac{1}{2},$

as desired.

· R Pho · ki 3. Bill plays a game in which he rolls two fair standard six-sided dice with sides labeled one 12 Yes through six. He wins if the number on one of the dice is three times the number on the other die. If Bill plays this game three times, compute the probability that he wins at least once.

Solution: The answer is $\frac{217}{729}$.

He has a $\frac{1}{9}$ chance of winning each game, and so his chance of winning at least once is $1 - (\frac{8}{9})^3$. Let $A = \frac{1}{9} + \frac{1}{9} + \frac{1}{7} + \frac{1}{9},$ 面对加根教林等限 mythill the the te

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A =
$$\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{9}$$
,
B = $\frac{1}{2 \cdot 3} + \frac{1}{2 \cdot 5} + \frac{1}{2 \cdot 9} + \frac{1}{3 \cdot 5} + \frac{1}{3 \cdot 9} + \frac{1}{5 \cdot 9}$,
C = $\frac{1}{2 \cdot 3 \cdot 5} + \frac{1}{2 \cdot 3 \cdot 9} + \frac{1}{2 \cdot 5 \cdot 9} + \frac{1}{3 \cdot 5 \cdot 9}$.

而时间的称林塔梯 Will the the B Compute the value of A + B + C.

Solution: The answer is $\frac{449}{720}$.

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 $1 + A + B + C + \frac{1}{270}$ factorizes to $(1 + \frac{1}{2})(1 + \frac{1}{3})(1 + \frac{1}{5})(1 + \frac{1}{9})$ and can be shown to equal $\frac{8}{3}$. So the answer is $\frac{8}{3} - 1 - \frac{1}{270} = \frac{449}{720}$. The state of the s Astitute ## # 18

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