

1. Interpret the regular expression.
  - (a) Strings with exactly one 1.
  - (b) Strings of even length.
  - (c) Strings in which 0s occur only in blocks of 3.
2. For these, especially part (c), equivalent answers may exist. Grade accordingly; compile a list of common correct answers as you go.
  - (a)  $\Sigma^*1\Sigma^*$
  - (b)  $0\Sigma^*0 \cup 1\Sigma^*1$
  - (c)  $\Sigma(0^*10^*10^*)^*$
3. Generating function basics
  - (a) Rewriting our sequence as a generating function gives  $1 + 0x + 3x^2 + 0x^3 + 9x^4 + \dots$ . We see that this is a geometric series starting with 1 with common ratio  $3x^2$ , so our generating function in closed form is  $\frac{1}{1-3x^2}$ .
  - (b) Each decomposition of an arbitrary integer  $n$  into 2 positive integers  $k + (n - k) = n$  contributes  $a_k b_{n-k}$  to the coefficient of  $x^n$ . Summing over all possible such decompositions gives us  $c_n = \sum_{k=0}^n a_k b_{n-k}$ .
  - (c) It is clear that  $\frac{1}{1-x^c}$  is the generating function for  $c_n$ , the number of ways to make change with coins of value  $c$  cents. The generating function we're looking at is the product of the generating functions with  $c \in \{1, 5, 10, 25\}$ . By part (b), we see that the sequence for  $\frac{1}{(1-x^{c_1})(1-x^{c_2})}$  is given by  $(c_1 c_2)_n = \sum_{k=0}^n a_k b_{n-k}$ . This is indeed the number of ways to make change for  $n$  cents given coins of value  $c_1$  and  $c_2$ .
4. Generating functions and recurrence relations
  - (a) Let  $f(x)$  be the generating function for the Fibonacci sequence. Then  $f(x)$  satisfies  $f(x) - xf(x) - x^2f(x) = (F_1 - F_0)x + F_0$ . Thus  $f(x) = \frac{1}{1-x-x^2}$ .
  - (b) Letting  $f(x) = \frac{1+x^2}{1+2x-x^3}$ ,  $f(x) + 2xf(x) - x^3f(x) = 1 + x^2$ . Then coefficients of terms of degree  $n \geq 3$  on the LHS must sum to zero, i.e.  $F_n + 2F_{n-1} - F_{n-3} = 0$ , which gives the desired recurrence.
  - (c) We know that  $(\sum a_n x^n)(c_0 + c_1 x + \dots + c_k x^k) = p(x)$ . For  $n$  sufficiently large ( $n > d$ ),  $0x^n = (a_n c_0 + a_{n-1} c_1 + \dots + a_{n-k} c_k)x^n$ . Therefore,  $0 = a_n c_0 + a_{n-1} c_1 + \dots + a_{n-k} c_k$ .
5. Let the generating function representing  $A$  be  $f_A(x) = a_0 + a_1 x + a_2 x^2 + \dots$  and  $B$  be  $f_B(x) = b_0 + b_1 x + b_2 x^2 + \dots$ . Then the number of strings of length  $n$  in  $A \circ B$  is the number of strings of length  $n$  formed from strings of length 0 in  $A$  and length  $n$  in  $B$ , the strings of length 1 in  $A$  and those of length  $n - 1$  in  $B$ , etc. so we see that  $c_n$  must be the sum

$$c_n = \sum_{k=0}^n a_k b_{n-k}$$

and thus our generating function for  $A \circ B$  must be the product of the generating functions for  $A$  and  $B$ ,  $f_A(x) \cdot f_B(x)$ .

6. As above, let the generating function representing  $A$  be  $f_A(x) = a_0 + a_1 x + a_2 x^2 + \dots$  and  $B$  be  $f_B(x) = b_0 + b_1 x + b_2 x^2 + \dots$ . Consider the number of strings of length  $n$  in  $A \cup B$ . We can either have a string of length  $n$  from  $f_A(x)$ , of which we have  $a_n$ , or a string of length  $n$  from  $B$ , of which we have  $b_n$ . So we have  $a_n + b_n$  ways to have a string of length  $n$  in our regular expression  $A \cup B$ , so our generating function must be  $f_A(x) + f_B(x)$ .

7. For any regular expression  $A$ ,  $A^*$  consists of all of the product sets of elements in  $A, A^0, A^1, A^2, \dots$ . We claim that the generating function of  $A^n$  is  $[f_A(x)]^n$ , where  $f_A(x)$  is the generating function for  $A$ . We proceed by induction. Clearly the generating function of  $A$  is  $f_A(x) = [f_A(x)]^1$ , so this establishes our base case. Now assume for some  $n \geq 1$  the statement holds:  $[f_A(x)]^n$  is the generating function for  $A^n$ . Now consider the generating function for  $A_{n+1}$ : we can think of this as the concatenation of regular expressions  $A_n$  and  $A$ . By problem 6, this means our generating function must be  $[f_A(x)]^{n+1}$ . So our generating function in total is the union of all the generating functions for  $A^i$ ; by problem 5 this is

$$1 + f_A(x) + [f_A(x)]^2 + [f_A(x)]^3 + \dots$$

This a geometric series with first term 1 and common ratio  $f_A(x)$ , so our generating function for  $A^*$  is

$$\frac{1}{1 - f_A(x)}$$

8. Strings in which 0s appear only in odd blocks and 1s appear only in blocks of size 1 or 2.  
9. Using the rules developed in (5-7), we can calculate the generating function to be

$$(1 + x + x^2) \frac{1}{1 - \left(\frac{1}{1-x^2}\right)x(x+x^2)} \left(1 + \left(\frac{1}{1-x^2}\right)x\right) = \frac{x^4 - x^2 - 2x - 1}{x^3 + 2x^2 - 1}$$

10. From 4(c) and the closed form found in 9, we know that  $a_n$  satisfies the recurrence relation:

$$-1a_n + 2a_{n-2} + a_{n-3} = 0$$