Caltech Harvey Mudd Mathematics Competition

Team Round **November 23, 2013**

1. In how many ways can you rearrange the letters of 'Alejandro' such that it contains one of the words 'ned' or 'den'?

2. Two circles of radii 7 and 17 have a distance of 25 between their centers. What is the difference between the lengths of their common internal and external tangents (positive difference)?

3. Let p_n be the product of the nth roots of 1. For integral $x > 4$, let $f(x) = p_1 - p_2 + p_3 - p_4 + \cdots + (-1)^{x+1} p_x$. What is $f(2010)$?

4. The numbers 25 and 76 have the property that when squared in base 10, their squares also end in the same two digits. A positive integer that has at most 3 digits when expressed in base 21 and also has the property that its base 21 square ends in the same 3 digits is called amazing. Find the sum of all amazing numbers. Express your answer in base 21.

5. Compute the number of lattice points bounded by the quadrilateral formed by the points $(0, 0)$, $(0, 140)$, $(140, 0)$, and $(100, 100)$ (including the quadrilateral itself). A lattice point on the x-y plane is a point (x, y), where both x and y are integers.

6. Let $a_1 < a_2 < a_3 < ... < a_n < ...$ be positive integers such that, for $n = 1, 2, 3, ...$,
$$a_{2n} = a_n + n.$$
Given that if a_n is prime, then n is also, find a_{2014}.

7. The points $(0, 0)$, $(a, 5)$, and $(b, 11)$ are the vertices of an equilateral triangle. Find ab.

8. Two kids A and B play a game as follows: from a box containing n marbles ($n > 1$), they alternately take some marbles for themselves, such that:
    1. A goes first.
    2. The number of marbles taken by A in his first turn, denoted by k, must be between 1 and $n - 1$, inclusive.
    3. The number of marbles taken in a turn by any player must be between 1 and k, inclusive.

    The winner is the one who takes the last marble. Determine all natural numbers n for which A has a winning strategy.

9. A 7×7 grid of unit-length squares is given. Twenty-four 1×2 dominoes are placed in the grid, each covering two whole squares and in total leaving one empty space. It is allowed to take a domino adjacent to the empty square and slide it lengthwise to fill the whole square, leaving a new one empty and resulting in a different configuration of dominoes. Given an initial configuration of dominoes for which the maximum possible number of distinct configurations can be reached through any number of slides, compute the maximum number of distinct configurations.

10. Compute the lowest positive integer k such that none of the numbers in the sequence $\{1, 1 + k, 1 + k + k^2, 1 + k + k^2 + k^3, \cdots\}$ are prime.