Individual Round | **Caltech Harvey Mudd Mathematics Competition** | **November 23, 2013**

1. Compute
$$\sqrt{(\sqrt{63}+\sqrt{112}+\sqrt{175})(-\sqrt{63}+\sqrt{112}+\sqrt{175})(\sqrt{63}-\sqrt{112}+\sqrt{175})(\sqrt{63}+\sqrt{112}-\sqrt{175})}.$$

2. Consider the set $S = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. How many distinct 3-element subsets are there such that the sum of the elements in each subset is divisible by 3?

3. Let a^2 and b^2 be two integers. Consider the triangle with one vertex at the origin, and the other two at the intersections of the circle $x^2 + y^2 = a^2 + b^2$ with the graph $ay = b|x|$. If the area of the triangle is numerically equal to the radius of the circle, what is this area?

4. Suppose $f(x) = x^3 + x - 1$ has roots α, β, and γ. What is
$$\frac{\alpha^3}{1-\alpha}+\frac{\beta^3}{1-\beta}+\frac{\gamma^3}{1-\gamma}?$$

5. Lisa has a 2D rectangular box that is 48 units long and 126 units wide. She shines a laser beam into the box through one of the corners such that the beam is at a 45° angle with respect to the sides of the box. Whenever the laser beam hits a side of the box, it is reflected perfectly, again at a 45° angle. Compute the distance the laser beam travels until it hits one of the four corners of the box.

6. How many ways can we form a group with an odd number of members (plural) from 99 people total? Express your answer in the form $a^b + c$, where a, b, and c are integers, and a is prime.

7. Let
$$S = \log_2 9 \log_3 16 \log_4 25 \cdots \log_{999} 1000000.$$
Compute the greatest integer less than or equal to $\log_2 S$.

8. A prison, housing exactly four hundred prisoners in four hundred cells numbered 1-400, has a really messed-up warden. One night, when all the prisoners are asleep and all of their doors are locked, the warden toggles the locks on all of their doors (that is, if the door is locked, he unlocks the door, and if the door is unlocked, he locks it again), starting at door 1 and ending at door 400. The warden then toggles the lock on every other door starting at door 2 (2, 4, 6, etc). After he has toggled the lock on every other door, the warden then toggles every third door (doors 3, 6, 9, etc.), then every fourth door, etc., finishing by toggling every 400th door (consisting of only the 400th door). He then collapses in exhaustion.

   Compute the number of prisoners who go free (that is, the number of unlocked doors) when they wake up the next morning.

9. Let A and B be fixed points on a 2-dimensional plane with distance $AB = 1$. An ant walks on a straight line from point A to some point C on the same plane and finds that the distance from itself to B always decreases at any time during this walk. Compute the area of the locus of points where point C could possibly be located.

10. A robot starts in the bottom left corner of a 4×4 grid of squares. How many ways can it travel to each square exactly once and then return to its start if it is only allowed to move to an adjacent (not diagonal) square at each step?

11. Assuming real values for p, q, r, and s, the equation

$$x^4 + px^3 + qx^2 + rx + s$$

has four non-real roots. The sum of two of these roots is $4 + 7i$, and the product of the other two roots is $3 - 4i$. Find q.

12. A cube is inscribed in a right circular cone such that one face of the cube lies on the base of the cone. If the ratio of the height of the cone to the radius of the cone is $2 : 1$, what fraction of the cone's volume does the cube take up? Express your answer in simplest radical form.

13. Let

$$y = \cfrac{1}{1 + \cfrac{1}{9 + \cfrac{1}{5 + \cfrac{1}{9 + \cfrac{1}{5 + \cdots}}}}}.$$

If y can be represented as $\dfrac{a\sqrt{b} + c}{d}$, where b is not divisible by the square of any prime, and the greatest common divisor of a and d is 1, find the sum $a + b + c + d$.

14. Alice wants to paint each face of an octahedron either red or blue. She can paint any number of faces a particular color, including zero. Compute the number of ways in which she can do this. Two ways of painting the octahedron are considered the same if you can rotate the octahedron to get from one to the other.

15. Find n in the equation

$$133^5 + 110^5 + 84^5 + 27^5 = n^5,$$

where n is an integer less than 170.