Caltech Harvey Mudd Mathematics Competition

Tiebreaker Round **March 3, 2012**

1. Let a_k be the number of ordered 10-tuples $(x_1, x_2, \ldots x_{10})$ of nonnegative integers such that

$$x_1^2 + x_2^2 + \cdots + x_{10}^2 = k.$$

Let $b_k = 0$ if a_k is even and $b_k = 1$ if a_k is odd. Find $\sum_{i=1}^{2012} b_{4i}$.

2. A convex octahedron in Cartesian space contains the origin in its interior. Two of its vertices are on the x-axis, two are on the y-axis, and two are on the z-axis. One triangular face F has side lengths $\sqrt{17}, \sqrt{37}, \sqrt{52}$. A second triangular face F' has side lengths $\sqrt{13}, \sqrt{29}, \sqrt{34}$. What is the minimum possible volume of the octahedron?

3. Three different faces of a regular dodecahedron are selected at random and painted. What is the probability that there is at least one pair of painted faces that share an edge?

4. The expression below has six empty boxes. Each box is to be filled in with a number from 1 to 6, where all six numbers are used exactly once, and then the expression is evaluated. What is the maximum possible final result that can be achieved?

$$\frac{\frac{\square}{\square} + \frac{\square}{\square}}{\frac{\square}{\square}}$$