Fall 2012 Caltech-Harvey Mudd Math Competition

November 17, 2012

Team Round

The team round will last for **75 minutes**, plus a five minute reading period at the beginning. The test will have two equally weighted parts: a power question and a short answer part. Teams will be able to work on the two parts simultaneously.

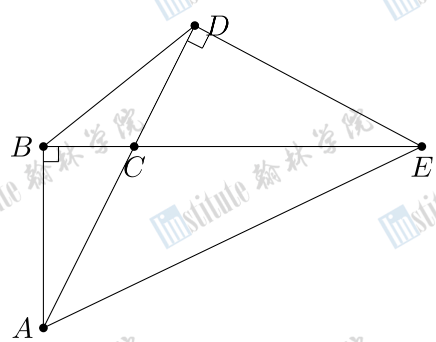
- During the five minute reading period, team members may not write. However, they may discuss the problems with each other.
- Teams may use blackboards or whiteboards.
- The power question will be similar in style to the ARML power round, although it will be shorter. Teams will be expected to fully justify their answers to the power question. The power question will be worth 90 points.
- The short answer part of the team round will have ten questions worth 9 points each with numerical or algebraic answers. In the short answer part, teams will not need to justify their answers. To get full credit, they will need only to write down the correct answer.

TR1. Find the remainder when 5^{2012} is divided by 3.

TR2. Consider a triangle ABC with points D on AB, E on BC, and let F be the intersection of AE and CD. Suppose $AD = 1, DB = 2, BE = 1, EC = 3$, and $CA = 5$. Find the value of the area of ECF minus the area of ADF.

TR3. A particular graph has 6 vertices, 12 edges, and has the property that it contains no Eulerian path; a Eulerian path is a route from vertex to vertex along edges that traces each edge exactly once. Determine all the possible degrees of its vertices in no particular order. There are two solutions, and you need to get both to get credit for this problem.

TR4. Consider the figure below, not drawn to scale.

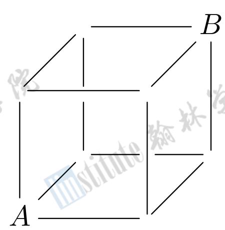


In this figure, assume that $AB \perp BE$ and $AD \perp DE$. Also, let $AB = \sqrt{6}$ and $\angle BED = \frac{\pi}{6}$. Find AC.

TR5. At each step, a rectangular tile of length 1, 2, or, 3 is chosen at random, what is the probability that the total length is 10 after 5 steps?

TR6. Suppose you have ten pairs of red socks, ten pairs of blue socks, and ten pairs of green socks in your drawer. You need to go to a party soon, but the power is currently off in your room. It is completely dark, so you cannot see any colors and unfortunately the socks are identically shaped. What is the minimum number of socks you need to take from the drawer in order to guarantee that you have at least one pair of socks whose colors match?

TR7. Consider a 1 by 2 by 3 rectangular prism. Find the length of the shortest path between opposite corners A and B that does not leave the surface of the prism.



TR8. Find the sum of all positive 30-digit palindromes. The leading digit is not allowed to be 0.

TR9. For a positive integer n, let $f(n)$ be equal to n if there is an integer x such that $x^2 - n$ is divisible by 2^{12}, and let $f(n)$ be 0 otherwise. Determine the remainder when

$$\sum_{n=0}^{2^{12}-1} f(n)$$

is divided by 2^{12}.

TR10. Let

$$N = \binom{2^{2012}}{0}\binom{2^{2012}}{1}\binom{2^{2012}}{2}\binom{2^{2012}}{3}\cdots\binom{2^{2012}}{2^{2012}}.$$

Let M be the number of 0's when N is written in binary. How many 0's does M have when written in binary? (Warning: this question is *very* hard.)