Fall 2012 Caltech-Harvey Mudd Math Competition

November 17, 2012

Mixer Round

In the mixer round, students will be grouped with students from other teams. This round will not count towards the students' or teams' scores, but there will be a separate small prize for the winning team of the mixer round.

MR1. Prove that $x = 2$ is the only real number satisfying $3^x + 4^x = 5^x$.

MR2. Show that

$$\sqrt{9+4\sqrt{5}} - \sqrt{9-4\sqrt{5}}$$

is an integer.

MR3. Two players A and B play a game on a round table. Each time they take turn placing a round coin on the table. The coin has a uniform size, and this size is at least 10 times smaller than the table size. They cannot place the coin on top of any part of other coins, and the whole coin must be on the table. If a player cannot place a coin, he loses. Suppose A starts first. If both of them plan their moves wisely, there will be one person who will always win. Determine whether A or B will win, and then determine his winning strategy.

MR4. Suppose you are given 4 pegs arranged in a square on a board. A "move" consists of picking up a peg, reflecting it through any other peg, and placing it down on the board. For how many integers $1 \le n \le 2013$ is it possible to arrange the 4 pegs into a *larger* square using exactly n moves? Justify your answers.

MR5. Find smallest positive integer that has a remainder of 1 when divided by 2, a remainder of 2 when divided by 3, a remainder of 3 when divided by 5, and a remainder of 5 when divided by 7.

MR6. Find the value of

$$\sum_{m|496, m>0} \frac{1}{m},$$

where $m|496$ means 496 is divisible by m.

MR7. What is the value of

$$\binom{100}{0} + \binom{100}{4} + \binom{100}{8} + \cdots + \binom{100}{100}?$$

MR8. An n-term sequence $a_0, a_1, \ldots, a_{n-1}$ will be called *sweet* if, for each $0 \le i \le n-1$, a_i is the number of times that the number i appears in the sequence. For example, $1, 2, 1, 0$ is a sweet sequence with 4 terms. Given that $a_0, a_1, \cdots, a_{2013}$ is a sweet sequence, find the value of $a_0^2 + a_1^2 + \cdots + a_{2013}^2$.