Caltech Harvey Mudd Mathematics Competition

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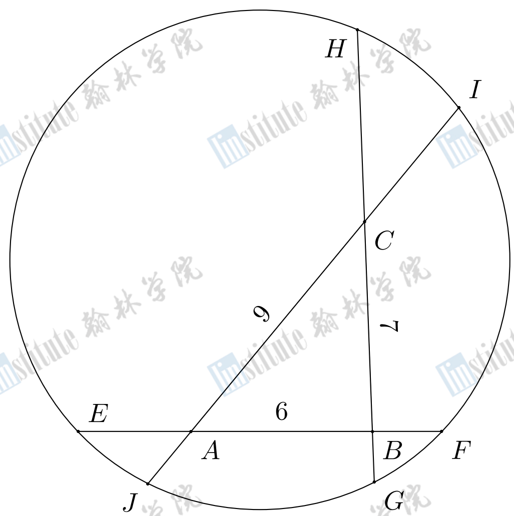
1. The monic polynomial f has rational coefficients and is irreducible over the rational numbers. If $f\left(\sqrt{5}+\sqrt{2}\right)=0$, compute $f\left(f\left(\sqrt{5}-\sqrt{2}\right)\right)$. (A polynomial is *monic* if its leading coefficient is 1. A polynomial is *irreducible* over the rational numbers if it cannot be expressed as a product of two polynomials with rational coefficients of positive degree. For example, x^2-2 is irreducible, but $x^2-1=(x+1)(x-1)$ is not.)

   Solution: Let $x=\sqrt{5}+\sqrt{2}$. Then $x^2=5+2+2\sqrt{10}$, so $(x^2-7)^2-40=0$. Thus $f(x)=(x^2-7)^2-40$ is a monic polynomial such that $f(\sqrt{5}+\sqrt{2})=0$. One can notice that if $y=\sqrt{5}-\sqrt{2}$, then $y^2=7-2\sqrt{10}$, so $0=(y^2-7)^2-40=f(y)$. Thus $f(\sqrt{5}-\sqrt{2})=0$, and so $f(f(\sqrt{5}-\sqrt{2}))=f(0)=(0^2-7)^2-40=\boxed{9}$.

   There are a several ways to check that f is irreducible. If we could factor f as a product of polynomials of positive degree with rational coefficients, then one of the factors would be a linear or quadratic polynomial. We can notice that the four roots of f are $\sqrt{5}+\sqrt{2}$, $\sqrt{5}-\sqrt{2}$, $-\sqrt{5}+\sqrt{2}$, and $-\sqrt{5}-\sqrt{2}$. Each of these roots is irrational, so it can't be the root of a linear polynomial with rational coefficients. It is also not hard to check that none of these roots are roots of a quadratic polynomial with rational coefficients, so we get a contradiction.

2. In the following diagram, points E, F, G, H, I, and J lie on a circle. The triangle ABC has side lengths $AB=6$, $BC=7$, and $CA=9$. The three chords have lengths $EF=12$, $GH=15$, and $IJ=16$. Compute $6\cdot AE+7\cdot BG+9\cdot CI$.



   Solution: We use the Power of a Point Theorem three times at A, B, and C to obtain the equations $AE\cdot AF=AI\cdot AJ$, $BG\cdot BH=BE\cdot BF$, and $CI\cdot CJ=CG\cdot CH$. Since we know the lengths of the chords, we can rewrite these equations just in terms of AE, BG, and CI:

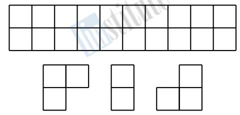
$$AE(12-AE)=(9+CI)(7-CI)$$
$$BG(15-BG)=(6+AE)(6-AE)$$
$$CI(16-CI)=(7+BG)(8-BG)$$

   Adding these three equations together and simplifying, we find that

$$12AE+15BG+16CI-(AE)^2-(BG)^2-(CI)^2=155-2CI+BG-(AE)^2-(BG)^2-(CI)^2$$

   We conclude that $12AE+14BG+18CI=155$, so $6AE+7BG+9CI=\boxed{\frac{155}{2}}$.

3. Compute the number of ways of tiling the 2×10 grid below with the three tiles shown. There is an infinite supply of each tile, and rotating or reflecting the tiles is not allowed.



Solution: Call the three tiles a Γ-tile, an I-tile, and a J-tile, respectively. It is easy to see that each Γ-tile must be paired with a J-tile to create a 2×3 rectangle. Thus we'd like to tile a 2×10 rectangle with 2×3 rectangles and 2×1 rectangles. We can therefore reduce the problem to tiling a 1×10 rectangle with 1×3 rectangles and 1×1 squares.

We can compute the number of ways to tile this rectangle using recursion. Let T_n be the number of tiling a $1 \times n$ rectangle with 1×3 and 1×1 tiles. We can tile a $1 \times n$ rectangle by first placing either a 1×1 or a 1×3 tile on the left. If we place a 1×1 tile, then the number of ways of tiling the remaining $n - 1$ squares is T_{n-1}. If we place a 1×3 tile, then the number of ways of tiling the remaining $n - 3$ squares is T_{n-3}. Thus $T_n = T_{n-1} + T_{n-3}$. Using $T_0 = T_1 = T_2 = 1$, we can use this recursive formula to compute T_n:

n	0	1	2	3	4	5	6	7	8	9	10
T_n	1	1	1	2	3	4	6	9	13	19	28

Thus there are $\boxed{28}$ ways of tiling the rectangle.

4. Compute the number of positive divisors of 2010.

Solution: We can factor $2010 = 2 \cdot 3 \cdot 5 \cdot 67$. A divisor of 2010 is the product of a subset of $\{2, 3, 5, 67\}$. There are $2^4 = 16$ such subsets, so 2010 has $\boxed{16}$ divisors.