Caltech Harvey Mudd Mathematics Competition

Team Round Solutions **February 20, 2010**

1. (*Yasha Berchenko-Kogan*) A matrix M is called *idempotent* if $M^2 = M$. Find an idempotent 2×2 matrix with distinct rational entries or write "none" if none exist.

Solution: Let $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Then $M^2 = \left(\begin{smallmatrix} a^2+bc & b(a+d) \\ c(a+d) & d^2+bc \end{smallmatrix}\right)$. We see that M is idempotent if and only if the following four equations are satisfied:

$$a = a^2 + bc \qquad b = b(a+d) \qquad c = c(a+d) \qquad d = d^2 + bc$$

In particular, from the second and third equations we see that either $a + d = 1$ or $b = c = 0$. Since the problems asks for M to have distinct entries, we cannot have $b = c = 0$, so we must have $a + d = 1$. We can write the first equation as $bc = a - a^2 = a(1-a) = ad$. Likewise, the fourth equation is also equivalent to $bc = d - d^2 = d(1-d) = ad$. Therefore, we are looking for distinct rational numbers a, b, c, d, such that $bc = ad$. One such example is $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} 2 & -1 \\ 6 & -3 \end{smallmatrix}\right)$. Thus, the answer is $\boxed{\text{any matrix } M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \text{ with distinct rational entries that satisfies } ad = bc \text{ and } a + d = 1.}$

2. (*Brian Lawrence*) The largest prime factor of $199^4 + 4$ has four digits. Compute the second largest prime factor.

Solution: We use the factoring trick $x^4 + 4y^4 = (x^2 + 2xy + 2y^2)(x^2 - 2xy + 2y^2)$, so $199^4 + 4 = (199^2+2\cdot199+2)(199^2-2\cdot199+2) = ((199+1)^2+1)((199-1)^2+1) = 40001\cdot39205$. Since $40001 = 4\cdot10^4+1$, we can use this factoring trick again to find that $40001 = (2\cdot10^2+2\cdot10+1)(2\cdot10^2-2\cdot10+1) = 221\cdot181$. Thus the 4-digit prime factor cannot divide 40001, so it must divide $39205 = 5\cdot7841$. We can check that 7841 is not divisible by 2, 3, 5, or 7. We know that 7841 cannot be divisible by any larger number, because then 39205 would not have a 4-digit prime factor. Thus 7841 must be the largest prime factor of $199^4 + 4$. We check that $221 = 15^2 - 2^2 = 17\cdot13$ and that 181 is prime. Thus the prime factorization of $199^4 + 4$ is $5\cdot13\cdot17\cdot181\cdot7841$, and so the second largest prime factor is $\boxed{181}$.

3. (*Yasha Berchenko-Kogan*) Assume that the earth is a perfect sphere. A plane flies between $30°N\, 45°W$ and $30°N\, 45°E$ along the shortest possible route. Let θ be the northernmost latitude that the plane flies over. Compute $\sin\theta$.

Solution: For simplicity, we can let the radius of the sphere be 1. If z is the vertical axis, we see that the plane takes off and lands at $z = \sin 30° = \frac{1}{2}$. Since the plane's origin and destination are 90° of longditude apart, we can choose coordinates so that the plane's starting point is in the xz-plane and destination is in the yz-plane. Since $\cos 30° = \frac{\sqrt{3}}{2}$, we see that the planes starts at $\left(\frac{\sqrt{3}}{2}, 0, \frac{1}{2}\right)$ and flies to $\left(0, \frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

The shortest route between two points on a sphere is along a great circle. Any great circle is the intersection of the sphere with a plane through the origin. There is a unique plane through $(0,0,0)$, $\left(\frac{\sqrt{3}}{2}, 0, \frac{1}{2}\right)$, and $\left(0, \frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. A quick computation will show that the equation of the plane is $x+y+\sqrt{3}z = 0$.

By symmetry, the northernmost point on the great circle will be when $x = y$. Thus we must solve the equations $2x + \sqrt{3}z = 0$ along with the equation of the sphere $1 = x^2 + y^2 + z^2 = 2x^2 + z^2$. Substituting $x = -\frac{\sqrt{3}}{2}z$, we see that $1 = 2\cdot\frac{3}{4}z^2 + z^2 = \frac{5}{2}z^2$. Notice that the z-coordinate of a point at a latitude of θ is $\sin\theta$. Thus $\sin\theta = z = \sqrt{\frac{2}{5}} = \boxed{\frac{\sqrt{10}}{5}}$.

4. (*Connor Ahlbach*) Compute the number of integer solutions (x, y) to $xy - 18x - 35y = 1890$.

Solution: We use a factoring trick:

$$(x-35)(y-18) = xy - 18x - 35y + 35\cdot18 = 1890 + 35\cdot18 = 1890 + 630 = 2520$$

For each pair of integers (a, b) with $ab = 2520$, we can find an integer solution (x, y) by setting $x = a+35$ and $y = b+18$. Conversely, each integer solution (x, y) corresponds to a pair of divisors of 2520. Thus we have reduced the problem to finding the number of divisors (both positive and negative) of 2520.

We can factor $2520 = 2^3 \cdot 3^2 \cdot 5 \cdot 7$. Using the formula for the number of positive divisors of an integer, we see that 2520 has $(3+1)(2+1)(1+1)(1+1) = 48$ positive divisors. The negative divisors of 2520 are just the opposites of the positive divisors, so there are 48 of them. Thus there are 96 positive or negative divisors of 2520, and so the equation has $\boxed{96}$ solutions.

5. (*Yasha Berchenko-Kogan*) The *popularity* of a positive integer n is the number of positive integer divisors of n. For example, 1 has popularity 1, and 12 has popularity 6. For each number n between 1 and 30 inclusive, Cathy writes the number n on k pieces of paper, where k is the popularity of n. Cathy then picks a piece of paper at random. Compute the probability that she will pick an even integer.

Solution: This problem is a straightforward computation, but there are some shortcuts that can let us solve it faster. Let $\tau(n)$ denote the number of positive divisors of n. If n is odd, then $\tau(2^k n) = (k+1)n$. This fact allows us to compute $\tau(n)$ for all n less than or equal to 30 faster by making the following table.

n	$\tau(n)$	$2n$	$\tau(2n)$	$4n$	$\tau(4n)$	$8n$	$\tau(8n)$	$16n$	$\tau(16n)$
1	1	2	2	4	3	8	4	16	5
3	2	6	4	12	6	24	8		
5	2	10	4	20	6				
7	2	14	4	28	6				
9	3	18	6						
11	2	22	4						
13	2	26	4						
15	4	30	8						
17	2								
19	2								
21	4								
23	2								
25	3								
27	4								
29	2								
	37		36		21		12		5

There are shortcuts to adding the numbers in each column. Notice that for most odd n in the table, $\tau(n) = 2$. Thus one can quickly add up the numbers in the first column by computing $2 \cdot 15 = 30$, and then adding the difference between each entry and 2 to get a total of 37. A similar trick can be used for adding up the values of $\tau(2n)$.

Thus Cathy has 37 pieces of paper with an odd number written on them, and she has $37+36+21+12+5 = 111$ pieces of paper in total, so the probability of picking an even integer is $1 - \frac{37}{111} = \boxed{\frac{2}{3}}$.

6. (*Yasha Berchenko-Kogan*) Zach rolls five tetrahedral dice, each of whose faces are labeled 1, 2, 3, and 4. Compute the probability that the sum of the values of the faces that the dice land on is divisible by 3.

Solution: The easiest way to do this problem is to use generating functions. If $f(x) = \frac{1}{4}(x+x^2+x^3+x^4)$, then the nth coefficient of f is the probability that the outcome of a single dice roll is n. Let $g(x) = (f(x))^5$. It is not hard to check that the nth coefficient of g is the probability that the sum of five dice rolls is n. Let $\omega = \frac{-1+i\sqrt{3}}{2}$ be a third root of unity. Notice that $1^n + \omega^n + (\omega^2)^n$ is equal to zero if n is not divisible by 3 and is equal to 3 if n is divisible by 3. We conclude that $\frac{1}{3}(g(1) + g(\omega) + g(\omega^2))$ is the sum of the nth coefficients of g where n is divisible by 3, which is precisely the probability that the sum of five dice rolls is divisible by 3. Thus, this probability is

$$\begin{aligned}
\frac{1}{3}\left(g(1)+g(\omega)+g(\omega^2)\right) &= \frac{1}{3}\left((f(1))^5+(f(\omega))^5+(f(\omega^2))^5\right) \\
&= \frac{1}{3}\left(1+\tfrac{1}{1024}(\omega+\omega^2+1+\omega)+\tfrac{1}{1024}(\omega^2+\omega+1+\omega^2)\right) \\
&= \frac{1}{3}\left(1+\tfrac{1}{1024}\omega+\tfrac{1}{1024}\omega^2\right) = \frac{1}{3}\left(1-\frac{1}{1024}\right) = \boxed{\frac{341}{1024}}
\end{aligned}$$

7. (*Yasha Berchenko-Kogan*) Compute all real numbers a such that the polynomial x^4+ax^3+1 has exactly one real root.

Solution: Let r be the real root of the polynomial. Since the polynomial must have an even number of nonreal roots, we know that r is either a double root or a quadruple root. If r were a quadruple root, then the polynomial would have the form $x^4+ax^3+1=(x-r)^4=x^4-4rx^3+6r^2x^2-4r^3x+r^4$. This would imply that $0=4r^3$, which implies that $r=0$, which contradicts the fact that $r^4=1$. Therefore, r must be a double root of x^4+ax^3+1.

Let the two complex roots of x^4+ax^3+1 be $c\pm di$ for some real numbers c and d. Then

$$\begin{aligned}
x^4+ax^3+1 &= (x-r)^2(x-(c+di))(x-(c-di)) = (x^2-2rx+r^2)(x^2-2cx+c^2+d^2) \\
&= x^4-2(r+c)x^3+(r^2+4rc+c^2+d^2)x^2-2(rc^2+rd^2+r^2c)x+r^2(c^2+d^2)
\end{aligned}$$

We thus have the following system of equations that we would like to solve for a:

$$\begin{aligned}
-2(r+c) &= a \\
r^2+4rc+c^2+d^2 &= 0 \\
-2(rc^2+rd^2+r^2c) &= 0 \\
r^2(c^2+d^2) &= 1
\end{aligned}$$

The fourth equation implies that $r\neq 0$, so we can factor $-2r$ out of the third equation to obtain $c^2+d^2+rc=0$. We conclude that $c^2+d^2=-rc$. Substituting this into the second equation, we see that $r^2+3rc=0$. Again, since $r\neq 0$, we find that $r+3c=0$, so $c=-\frac{r}{3}$. Substituting $c^2+d^2=-rc$ into the fourth equation, we see that $r^2(-rc)=1$, so $c=-\frac{1}{r^3}$. We conclude that $\frac{r}{3}=\frac{1}{r^3}$, so $r^4=3$, and so $r=\pm\sqrt[4]{3}$. We conclude that $a=-2(r+c)=-2(r-\frac{r}{3})=-\frac{4}{3}r=\boxed{\pm\frac{4\sqrt[4]{3}}{3}}$. It is not hard to check that both of these values of a do indeed give polynomials whose only real root is $r=\mp\sqrt[4]{3}$.

It is also possible to do this problem using calculus by noting that x^4+ax^3+1 and its derivative must have a common root.

8. (*Yasha Berchenko-Kogan*) Alice and Bob are going to play a game called extra tricky double rock paper scissors (ETDRPS). In ETDRPS, each player simultaneously selects *two* moves, one for his or her right hand, and one for his or her left hand. Whereas Alice can play rock, paper, or scissors, Bob is only allowed to play rock or scissors. After revealing their moves, the players compare right hands and left hands separately. Alice wins if she wins *strictly* more hands than Bob. Otherwise, Bob wins. For example, if Alice and Bob were to both play rock with their right hands and scissors with their left hands, then both hands would be tied, so Bob would win the game. However, if Alice were to instead play rock with both hands, then Alice would win the left hand. The right hand would still be tied, so Alice would win the game. Assuming both players play optimally, compute the probability that Alice will win the game.

Solution: We first introduce some shorthand. Let an expression like $B:RS$ mean "Bob plays rock with his left hand and scissors with his right hand,". Since Bob cannot play paper, Alice would never play scissors, because playing rock would always be strictly better. Notice that $B:SS$ beats $A:PP$,

$A : RP$, and $A : PR$, but loses to $A : RR$. On the other hand $B : RS$ beats $A : PP$ and $A : RP$, but loses to $A : PR$ and $A : RR$. Thus $B : SS$ is strictly a better move than $B : RS$. Similarly, $B : SS$ is strictly a better move than $B : SR$. Thus, if Bob plays rationally, he will only play $B : SS$ or $B : RR$. Now, notice that $A : PP$, $A : PR$, and $A : RP$ all beat $B : RR$ but lose to $B : SS$. Thus the moves $A : PP$, $A : PR$, and $A : RP$ are equivalent, so the game where Alice can only play $A : PP$ and $A : RR$ is equivalent to the original game. We see that $A : PP$ beats $B : RR$ but loses to $B : SS$, and $A : RR$ beats $B : SS$ but loses to $B : RR$. Thus, if both players play optimally, the game is symmetric, so the probability that Alice wins is $\boxed{\frac{1}{2}}$.

9. (*Yasha Berchenko-Kogan*) Compute the positive integer n such that $\log_3 n < \log_2 3 < \log_3(n + 1)$.

   Solution: We immediately see that $\log_3 3 \le 1 < \log_2 3 < 2 = \log_3 9$. Thus $3 \le n \le 8$. The trick for getting a better estimate is to multiply the inequality by an integer. For example, we know that $2\log_3 n < 2\log_2 3 < 2\log_3(n + 1)$, which is equivalent to $\log_3(n^2) < \log_2 9 < \log_3((n + 1)^2)$. We know that $\log_3(5^2) < \log_3 27 = 3 < \log_2 9 < 4 = \log_3(9^2)$. We thus see that $5 \le n \le 8$. Likewise, we know that $\log_3(n^3) < \log_2 27 < \log_3((n + 1)^3)$ and $\log_3(4^3) < \log_3 81 = 4 < \log_2 27 < 5 = \log_3 243 < \log_3(7^3)$, so $4 \le n \le 6$. Combined with our earlier knowledge, we see that $n = 5$ or $n = 6$. It remains to determine which of $\log_2 3$ and $\log_3 6$ is greater.

   To simplify our calculations, notice that $\log_3 6 = 1 + \log_3 2$. We note that $4\log_2 3 = \log_2 81$, which is between 6 and 7. Unfortunately, $4\log_3 6 = 4 + \log_3 16$ is also between 6 and 7, so we still don't know which one is bigger. Next we try computing $5\log_2 3 = \log_2 243$, which is between 7 and 8. We see that $5\log_3 6 = 5 + \log_3 32$ is between 8 and 9, so we can conclude that $\log_2 3 < \log_3 6$. Thus $\log_3 5 < \log_2 3 < \log_3 6$, so $\boxed{n = 5}$.

10. (*Brian Lawrence*) Compute the number of 10-bit sequences of 0's and 1's do not contain 001 as a subsequence.

    Solution: We'll deal with the sequence of all zeroes as a special case, so assume for now that all of the sequences have at least one 1. Notice that if we remove the trailing zeroes from a 10-bit sequence not containg 001, then we obtain a n-bit sequence ending in a 1 that does not contain two zeroes in a row, where $1 \le n \le 10$. Conversely, if we start with an n-bit sequence ending in a 1 that does not contain two zeroes in a row, then we can recover the original 10-bit sequence by adding zeroes at the end. Thus we have reduced the problem to counting the number of n-bit sequences that end in a 1 and do not have two zeroes in a row.

    Let S_n denote the number of n-bit sequences that do not contain two consecutive zeroes and end in a one. Consider such a sequence. If the first digit is a zero, then the second digit must be a one. The remaining $n - 2$ digits can be any sequence that does not contain two zeroes in a row and ends in a 1. On the other hand, if the first digit is a one, then the remaining $n - 1$ digits can be any sequence that does not contain two zeroes and ends in a one. Thus $S_n = S_{n-2} + S_{n-1}$. Notice that $S_1 = 1$ (the only possible one-bit sequence is 1), and $S_2 = 2$ (the only possible two-bit sequences are 01 and 11). Thus $S_n = F_{n+1}$, where F_{n+1} denotes the $n + 1$-st Fibonacci number.

    The total number of 10-bit sequences that do not contain 001 is the sum of S_n where n ranges from 1 to 10, plus one for the sequence of all zeroes that we ignored earlier. Thus, the answer is the sum $1 + F_2 + F_3 + \cdots + F_{11}$. Notice that $1 = F_1$, so our answer is the sum of the first eleven Fibonacci numbers. A well-known formula that can be easily proved with induction states that the sum of the first m Fibonacci numbers is $F_{m+2} - 1$, so our answer is $F_{13} - 1$. We can quickly compute that the first thirteen terms of the Fibonacci sequence are $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233$, so the answer is $233 - 1 = \boxed{232}$.