Caltech Harvey Mudd Mathematics Competition

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The *roots*, also called *zeroes*, of a function f are the values x such that $f(x) = 0$. You're familiar with the quadratic formula $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$, which computes the roots of a quadratic polynomial $ax^2 + bx + c$ in terms of a, b, and c. In this problem you will derive the cubic formula, which computes the roots of a cubic polynomial $ax^3 + bx^2 + cx + d$ with $a \neq 0$.

For this part of the contest, you must fully justify all of your answers unless otherwise specified. In your solutions, you may refer to the answers of earlier problems (but not later problems or later parts of the same problem), even if you were not able to solve those problems.

1. (a) Show that $\sqrt[3]{2} + \sqrt[3]{4}$ is a root of the polynomial $x^3 - 6x - 6$.

   Solution: We plug in $\sqrt[3]{2} + \sqrt[3]{4}$ for x in $x^3 - 6x - 6$, and expand using the binomial theorem.

$$\left(\sqrt[3]{2} + \sqrt[3]{4}\right)^3 - 6\left(\sqrt[3]{2} + \sqrt[3]{4}\right) - 6 = 2 + 3\sqrt[3]{16} + 3\sqrt[3]{32} + 4 - 6\sqrt[3]{2} - 6\sqrt[3]{4} - 6 = 0$$

   Thus $\sqrt[3]{2} + \sqrt[3]{4}$ is a root of $x^3 - 6x - 6$.

   (b) Show that $\sqrt[3]{u} + \sqrt[3]{v}$ is a root of the polynomial $x^3 - (3\sqrt[3]{u}\sqrt[3]{v})x - (u + v)$.

   Solution: Like in the previous part, we plug in $\sqrt[3]{u} + \sqrt[3]{v}$ into the polynomial.

$$\left(\sqrt[3]{u} + \sqrt[3]{v}\right)^3 - \left(3\sqrt[3]{u}\sqrt[3]{v}\right)\left(\sqrt[3]{u} + \sqrt[3]{v}\right) - (u+v) = \left(u + 3\sqrt[3]{u^2v} + 3\sqrt[3]{uv^2} + v\right) - \left(3\sqrt[3]{u^2v} + 3\sqrt[3]{uv^2}\right) - (u+v) = 0$$

   Thus $\sqrt[3]{u} + \sqrt[3]{v}$ is a root of $x^3 - 3\sqrt[3]{u}\sqrt[3]{v}x - (u + v)$.

2. (a) Using part 1b, find a real root of $x^3 - 12x - 34$.

   Solution: We'd like to use part 1b, so we set $12 = 3\sqrt[3]{u}\sqrt[3]{v}$ and $34 = u + v$ and try to solve for u and v. We know $uv = 64$ and $u + v = 34$. Using guess and check, we can find that $u = 2$ and $v = 32$ is a solution. Alternatively, we can compute $(v - u)^2 = (u + v)^2 - 4uv = 34^2 - 4 \cdot 64 = 30^2$. Thus we can set $v - u = 30$ and again conclude that $u = 2$ and $v = 32$. By part 1b, we know that $\sqrt[3]{2} + \sqrt[3]{32} = \sqrt[3]{2} + 2\sqrt[3]{4}$ is a root of $x^3 - 12x - 34$.

   (b) In the complex numbers, $x^3 - 12x - 34$ has three roots. Find the other two roots. (It might help to use the third root of unity $\omega = \frac{-1+i\sqrt{3}}{2}$ when expressing your answers.)

   Solution: In the complex numbers, 2 has three cube roots: $\sqrt[3]{2}$, $\omega\sqrt[3]{2}$, and $\omega^2\sqrt[3]{2}$, and similarly for 4. This suggests that trying something like $\omega\sqrt[3]{2} + \omega^2 2\sqrt[3]{4}$ would work. Indeed, we can use the binomial theorem and the fact that $\omega^3 = 1$ to find that

$$\left(\omega\sqrt[3]{2} + \omega^2 2\sqrt[3]{4}\right)^3 = \left(\omega\sqrt[3]{2}\right)^3 + 3\left(\omega\sqrt[3]{2}\right)\left(\omega^2 2\sqrt[3]{4}\right)\left(\omega\sqrt[3]{2} + \omega^2 2\sqrt[3]{4}\right) + \left(\omega^2 2\sqrt[3]{4}\right)^3 = 2 + 12\left(\omega\sqrt[3]{2} + \omega^2 2\sqrt[3]{4}\right) + 32$$

   Thus

$$\left(\omega\sqrt[3]{2} + \omega^2 2\sqrt[3]{4}\right)^3 - 12\left(\omega\sqrt[3]{2} + \omega^2 2\sqrt[3]{4}\right) - 34 = 0$$

   Thus $\omega\sqrt[3]{2} + \omega^2 2\sqrt[3]{4}$ is a root of the polynomial. The other root must be its complex conjugate $\omega^2\sqrt[3]{4} + \omega 2\sqrt[3]{4}$.

   Alternatively, we can factor the root we know out of the polynomial. We see that

$$x^3 - 12x - 34 = \left(x - \sqrt[3]{2} - 2\sqrt[3]{4}\right)\left(x^2 + \left(\sqrt[3]{2} + 2\sqrt[3]{4}\right)x + \left(8\sqrt[3]{2} + \sqrt[3]{4} - 4\right)\right)$$

Using the quadratic formula, we conclude that

$$\begin{aligned} x &= \frac{1}{2}\left(-\sqrt[3]{2}-2\sqrt[3]{4}\pm\sqrt{\left(8\sqrt[3]{2}+\sqrt[3]{4}+8\right)-4\left(8\sqrt[3]{2}+\sqrt[3]{4}-4\right)}\right)\\ &= \frac{1}{2}\left(-\sqrt[3]{2}-2\sqrt[3]{4}\pm i\sqrt{3}\sqrt{8\sqrt[3]{2}+\sqrt[3]{4}-8}\right)\\ &= \frac{1}{2}\left(-\sqrt[3]{2}-2\sqrt[3]{4}\pm i\sqrt{3}\left(\sqrt[3]{2}-2\sqrt[3]{4}\right)\right) \end{aligned}$$

The hint suggests that we write the roots in terms of ω, so we can write $x = \omega\sqrt[3]{2}+\omega^2 2\sqrt[3]{4}$ or $x = \omega^2\sqrt[3]{2}+\omega 2\sqrt[3]{4}$.

3. In part 1b, you found a root of $x^3-(3\sqrt[3]{u}\sqrt[3]{v})x-(u+v)$. Find the other two roots of this polynomial in the complex numbers in terms of $\sqrt[3]{u}$ and $\sqrt[3]{v}$.

   Solution: Using our intuition from problem 2b, we can guess that the roots should be $\omega\sqrt[3]{u}+\omega^2\sqrt[3]{v}$ and $\omega^2\sqrt[3]{u}+\omega\sqrt[3]{v}$. We can check this by plugging it into the polynomial:

$$\begin{aligned} &\left(\omega\sqrt[3]{u}+\omega^2\sqrt[3]{v}\right)^3-\left(3\sqrt[3]{u}\sqrt[3]{v}\right)\left(\omega\sqrt[3]{u}+\omega^2\sqrt[3]{v}\right)-(u+v)\\ &\qquad= \left(u+3\omega\sqrt[3]{u^2v}+3\omega^2\sqrt[3]{uv^2}+v\right)-\left(3\omega\sqrt[3]{u^2v}+3\omega^2\sqrt[3]{uv^2}\right)-(u+v)=0 \end{aligned}$$

   Thus $\omega\sqrt[3]{u}+\omega^2\sqrt[3]{v}$ is indeed a root of the polynomial. The third root must be its complex conjugate $\omega^2\sqrt[3]{u}+\omega\sqrt[3]{v}$.

4. Find all of the roots in the complex numbers of a polynomial of the form x^3+cx+d in terms of c and d.

   Solution: From earlier problems, we know how to solve polynomials of the form $x^3-(3\sqrt[3]{u}\sqrt[3]{v})x-(u+v)$, so we'd like to find u and v so that we get the polynomial x^3+cx+d. Thus we solve the equatations

$$\begin{aligned} c &= -3\sqrt[3]{u}\sqrt[3]{v}\\ d &= -(u+v) \end{aligned}$$

   We see that $uv = -\frac{c^3}{27}$ and $-(u+v)=d$. Using Vieta's formulas, we see that u and v are roots of the polynomial $y^2+dy-\frac{c^3}{27}$. Using the quadratic formula, we know that

$$u, v = \frac{1}{2}\left(-d\pm\sqrt{d^2+\frac{4c^3}{27}}\right)$$

   From our earlier work, we know that the three roots are $\sqrt[3]{u}+\sqrt[3]{v}$, $\omega\sqrt[3]{u}+\omega^2\sqrt[3]{v}$, and $\omega^2\sqrt[3]{u}+\omega\sqrt[3]{v}$. Substituting our values of u and v, we find that the three roots are

$$\sqrt[3]{\frac{1}{2}\left(-d+\sqrt{d^2+\frac{4}{27}c^3}\right)}+\sqrt[3]{\frac{1}{2}\left(-d-\sqrt{d^2+\frac{4}{27}c^3}\right)}$$

$$\omega\sqrt[3]{\frac{1}{2}\left(-d+\sqrt{d^2+\frac{4}{27}c^3}\right)}+\omega^2\sqrt[3]{\frac{1}{2}\left(-d-\sqrt{d^2+\frac{4}{27}c^3}\right)}$$

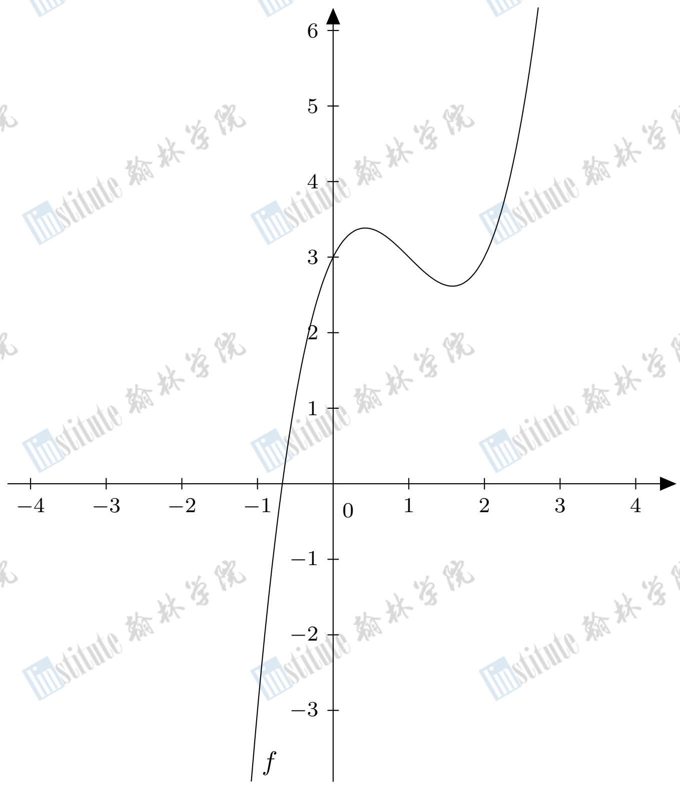
$$\omega^2\sqrt[3]{\frac{1}{2}\left(-d+\sqrt{d^2+\frac{4}{27}c^3}\right)}+\omega\sqrt[3]{\frac{1}{2}\left(-d-\sqrt{d^2+\frac{4}{27}c^3}\right)}$$

   However, there is a caveat. If $d^2+\frac{4}{27}c^3$ is negative, then u and v are not real, so the expression $\sqrt[3]{u}$ is not well-defined. (In the complex numbers, every nonzero number u has three cube roots, and we don't know which one to pick.) We can resolve this issue by choosing any cube root r with $r^3=u$ and defining $s=-\frac{c}{3r}$. Using the equation $c=-3\sqrt[3]{u}\sqrt[3]{v}$, we see that $s^3=v$, so s is the appropriate corresponding cube root of v.

5. Let $f(x) = x^3 - 3x^2 + 2x + 3$.

(a) Make a rough sketch of the graph of f. You do not need to justify your answer to this part.

Solution:



(b) Prove the graph of f is symmetric with respect to 180° degree rotations about some point P in the plane. Find the coordinates of P.

Solution: Notice that $f(0) = 3$, $f(1) = 3$, and $f(2) = 3$. This suggests that the inflection point of the cubic is at $(1, 3)$. To prove that the graph of f is symmetric about 180° rotations about P, we first notice that a 180° rotation about P sends a point (x, y) to $(2 - x, 6 - y)$. We must show that if (x, y) is on the cubic, then so is $(2 - x, 6 - y)$.

If (x, y) is on the cubic, then $y = x^3 - 3x^2 + 2x + 3$. We compute that

$$\begin{aligned} f(2-x) &= (2-x)^3 - 3(2-x)^2 + 2(2-x) + 3 \\ &= 8 - 12x + 6x^2 - x^3 - 12 + 12x - 3x^2 + 4 - 2x + 3 = -x^3 + 3x^2 - 2x + 3 = 6 - y \end{aligned}$$

Thus the point $(2 - x, 6 - y)$ is on the cubic.

(c) Let x_0 be the x-coordinate of P from part 5b. Show that $f(x + x_0)$ is a cubic polynomial whose x^2 coefficient is zero.

Solution: From the previous part, we know $x_0 = 1$. We compute

$$\begin{aligned} f(x + x_0) &= (x+1)^3 - 3(x+1)^2 + 2(x+1) + 3 \\ &= x^3 + 3x^2 + 3x + 1 - 3x^2 - 6x - 3 + 2x + 2 + 3 = x^3 - x + 3 \end{aligned}$$

The x^2 coefficient is zero, as desired.

6. Let $f(x) = x^3 + bx^2 + cx + d$.

   (a) Find a number x_0 in terms of b, c, and d, such that $f(x + x_0)$ is a cubic polynomial whose x^2 coefficient is zero.

   Solution: Let r_1, r_2, and r_3 be the three roots of $f(x)$. Notice that the roots of $f(x + x_0)$ are $r_1 - x_0$, $r_2 - x_0$, and $r_3 - x_0$. By Vieta's formulas, we know that $r_1 + r_2 + r_3 = -b$. Again, by Vieta's formulas, if the x^2 coefficient of $f(x+x_0)$ is zero, then $(r_1 - x_0) + (r_2 - x_0) + (r_3 - x_0) = 0$. We conclude that $3x_0 = r_1 + r_2 + r_3 = -b$, so $x_0 = -\frac{b}{3}$.

   (b) In the polynomial $f(x + x_0)$, find the x coefficient and the constant coefficient in terms of b, c, and d.

   Solution: We have $x_0 = -\frac{b}{3}$, so we compute

$$\begin{aligned} f(x - \tfrac{b}{3}) &= (x - \tfrac{b}{3})^3 + b(x - \tfrac{b}{3})^2 + c(x - \tfrac{b}{3}) + d \\ &= x^3 - bx^2 + \tfrac{1}{3}b^2x - \tfrac{1}{27}b^3 + bx^2 - \tfrac{2}{3}b^2x + \tfrac{1}{9}b^3 + cx - \tfrac{1}{3}bc + d \\ &= x^3 + \left(-\tfrac{1}{3}b^2 + c\right)x + \left(\tfrac{2}{27}b^3 - \tfrac{1}{3}bc + d\right) \end{aligned}$$

   Thus the x coefficient is $-\frac{1}{3}b^2 + c$, and the constant term is $\frac{2}{27}b^3 - \frac{1}{3}bc + d$.

7. Carefully explain how you would use the answers to the above problems to find all of the roots in the complex numbers of a polynomial of the form $ax^3 + bx^2 + cx + d$, assuming that $a \neq 0$. (You do not need to write down the general cubic formula to obtain full credit on this part.)

   Solution: Since $a \neq 0$, we know that $ax^3 + bx^2 + cx + d = 0$ if and only if $x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a} = 0$. Thus we can let $b' = \frac{b}{a}$, $c' = \frac{c}{a}$, and $d' = \frac{d}{a}$, and we can solve the equation $x^3 + b'x^2 + c'x + d' = 0$ to get the roots of f. Let $g(x) = x^3 + b'x^2 + c'x + d'$.

   We can let $x_0 = -\frac{b'}{3}$. By problem 6, we know that the x^2 coefficient of $g(x + x_0)$ is zero. We can let C be the x coefficient of $g(x + x_0)$, and we can let D be the constant term of $g(x + x_0)$, which we computed in problem 6. Thus $g(x + x_0) = x^3 + Cx + D$.

   Problem 4 lets us compute the roots r_1, r_2, and r_3 of $g(x + x_0)$. Thus $g(r_1 + x_0) = g(r_2 + x_0) = g(r_3 + x_0) = 0$. Thus the roots of f (which are the same as the roots of g) are $r_1 + x_0$, $r_2 + x_0$, and $r_3 + x_0$.