Caltech Harvey Mudd Mathematics Competition

1. (*Ying-Ying Tran*) Compute x such that $2009^{2010} \equiv x \pmod{2011}$ and $0 \le x < 2011$.

   Solution: We can check that 2011 is prime. By Fermat's Little Theorem, if $n \not\equiv 0 \pmod{2011}$ then $n^{2010} \equiv 1 \pmod{2011}$. In particular, $2009^{2010} \equiv \boxed{1} \pmod{2011}$.

2. (*Yasha Berchenko-Kogan*) Compute the number of "words" that can be formed by rearranging the letters of the word "syzygy" so that the y's are evenly spaced. (The y's are *evenly spaced* if the number of letters (possibly zero) between the first y and the second y is the same as the number of letters between the second y and the third y.)

   Solution: There are six ways of arranging the y's so that they are evenly spaced: `yyy???`, `?yyy??`, `??yyy?`, `???yyy`, `y?y?y?`, and `?y?y?y`. For each arrangement of the y's, there are six ways of placing the letters s, z, and g in the remaining space in the word. Thus, there are $\boxed{36}$ arrangements in total.

3. (*Yasha Berchenko-Kogan*) Let A and B be subsets of the integers, and let $A+B$ be the set containing all sums of the form $a+b$, where a is an element of A, and b is an element of B. For example, if $A = \{0, 4, 5\}$ and $B = \{-3, -1, 2, 6\}$, then $A + B = \{-3, -1, 1, 2, 3, 4, 6, 7, 10, 11\}$. If A has 1955 elements and B has 1891 elements, compute the smallest possible number of elements in $A + B$.

   Solution: It is easy to check that if $A = \{0, 1, 2, \dots, 1954\}$ and $B = \{0, 1, 2, \dots, 1890\}$, then $A + B = \{0, 1, 2 \dots, 3844\}$, which has $\boxed{3845}$ elements. It is easy to check that this is the smallest possible sumset possible: If $A = \{a_0, \dots, a_{1954}\}$ and $B = \{b_0, \dots, b_{1890}\}$ arranged in increasing order, then

   $$a_0 + b_0 < a_0 + b_1 < a_0 + b_2 < \cdots < a_0 + b_{1890} < a_1 + b_{1890} < a_2 + b_{1890} < \cdots < a_{1954} + b_{1890}$$

   Thus these 3845 elements of $A + B$ are distinct, so $|A + B| \ge 3845$.

   The numbers 1955 and 1891 are the years in which Harvey Mudd and Caltech were founded, respectively.

4. (*Yasha Berchenko-Kogan*) Compute the sum of all integers of the form p^n where p is a prime, $n \ge 3$, and $p^n \le 1000$.

   Solution: Since $n \ge 3$ and $p^n \le 1000$, we know that $p \le 10$, so $p \in \{2, 3, 5, 7\}$. Note that $2^9 < 1000 < 2^{10}$, $3^6 < 1000 < 3^7$, $5^4 < 1000 < 5^5$, and $7^3 < 1000 < 7^4$. Using the formula for a geometric series, we compute the desired sum:

   $$(2^3 + 2^4 + \cdots + 2^9) + (3^3 + 3^4 + 3^5 + 3^6) + (5^3 + 5^4) + (7^3) = 2^3 \frac{2^7 - 1}{2 - 1} + 3^3 \frac{3^4 - 1}{3 - 1} + 5^3 \cdot 6 + 343$$
   $$= 8 \cdot 127 + 27 \cdot 40 + 125 \cdot 6 + 343 = 1016 + 1080 + 750 + 343 = \boxed{3189}$$

5. (*Sam Elder*) In a season of interhouse athletics at Caltech, each of the eight houses plays each other house in a particular sport. Suppose one of the houses has a 1/3 chance of beating each other house. If the results of the games are independent, compute the probability that they win at least three games in a row.

   Solution: Let the probability that they don't win three in a row out of n games be p_n. Then $p_0 = p_1 = p_2 = 1$. For $n > 2$, consider the last games in their season. They can end with a loss, a loss and then a win, or a loss and then two wins. The probability each case occurs is $2p_{n-1}/3, 2p_{n-2}/9$, and $2p_{n-3}/27$. Thus, $p_n = 2(9p_{n-1} + 3p_{n-2} + p_{n-3})/27$, or $p_n \cdot 3^n = 2p_{n-1} \cdot 3^{n-1} + 2p_{n-2} \cdot 3^{n-2} + 2p_{n-3} \cdot 3^{n-3}$. Recursively calculating, we have $p_3 = 26/27, p_4 = 76/81, p_5 = 222/243, p_6 = 648/729$, and $p_7 = 1892/2187$, so $1 - p_7 = \boxed{295/2187}$ is the probability that they win three games in a row out of seven.

6. (*Yasha Berchenko-Kogan*) A positive integer n is *special* if there are exactly 2010 positive integers smaller than n and relatively prime to n. Compute the sum of all special numbers.

   Solution: If n is odd, then let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the prime factorization of n, where $\alpha_i \ge 1$. Let $\phi(n)$ denote the number of numbers less than or equal to n that are relatively prime to n. Using the formula for $\phi(n)$, we see that $\phi(n) = (p_1 - 1)p_1^{\alpha_1 - 1}(p_2 - 1)p_2^{\alpha_2 - 1} \cdots (p_k - 1)p_k^{\alpha_k - 1}$. Since n is odd, all of the p_i are odd, so all of the $p_i - 1$ are even, so $\phi(n)$ is divisible by 2^k. Since $\phi(n) = 2010 = 2 \cdot 3 \cdot 5 \cdot 67$, we conclude that $\phi(n)$ is divisible by only one factor of 2, so $k = 1$. Thus $2010 = (p_1 - 1)p_1^{\alpha_1 - 1}$. Note that if $\alpha_1 \geq 3$, then p_1^2 divides $2010 = 2 \cdot 3 \cdot 5 \cdot 67$, which is impossible. If $\alpha_1 = 2$, then p_1 divides 2010, so $p_1 \in \{3, 5, 67\}$, so $2010 = \phi(n) = (p_1 - 1)p_1^{2-1} \in \{6, 20, 66 \cdot 67\}$, which is a contradiction. Thus $\alpha_1 = 1$, so $2010 = p_1 - 1$. Therefore, $p_1 = 2011$, and so $n = p_1^{\alpha_1} = 2011$.

If n is even, then let $n = 2^{\alpha_0} p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$. We see that $\phi(n) = (2-1)2^{\alpha_0 - 1}(p_1 - 1)p_1^{\alpha_1 - 1}(p_2 - 1)p_2^{\alpha_2 - 1} \cdots (p_k - 1)p_k^{\alpha_k - 1} = 2^{\alpha_0 - 1}\phi\left(p_1^{\alpha_1} \cdots p_k^{\alpha_k}\right)$. Since 2010 is divisible by just one factor of 2, we know that α_0 is either 1 or 2. If $\alpha_0 = 2$, then $\phi\left(p_1^{\alpha_1} \cdots p_k^{\alpha_k}\right) = 1005$. Our earlier argument showed that 2^k must divide 1005, so $k = 0$, which is a contradiction. Thus $\alpha_0 = 1$ and $\phi\left(p_1^{\alpha_1} \cdots p_k^{\alpha_k}\right) = 2010$. We showed earlier that this implies that $p_1^{\alpha_1} \cdots p_k^{\alpha_k} = 2011$, so $n = 2 \cdot 2011 = 4022$.

Thus the only special numbers are 2011 and 4022, so their sum is $\boxed{6033}$.

7. (*Yasha Berchenko-Kogan*) Eight friends are playing informal games of ultimate frisbee. For each game, they split themselves up into two teams of four. They want to arrange the teams so that, at the end of the day, each pair of players has played at least one game on the same team. Determine the smallest number of games they need to play in order to achieve this.

Solution: They can play on a team with every other player in just three games. Let the players be A, B, C, D, E, F, G, and H. In the first game, players $ABCD$ play against $EFGH$. In the second game, players $ABEF$ play against $CDGH$. In the third game, players $ABGH$ play aginst players $CDEF$. We also need to check that it is impossible to achieve this in just two games. Since player A plays on a team with three other people each game, after the first two games, player A has played on a team with at most six other players, so he or she could not have played with all of them. Therefore, the friends must play $\boxed{3}$ games.

8. (*Yasha Berchenko-Kogan*) Compute the number of ways to choose five nonnegative integers a, b, c, d, and e, such that $a + b + c + d + e = 20$.

Solution: Given nonnegative integers a, b, c, d, and e that sum to 20, we construct a sequence of 20 dots and 4 dashes as follows: write a dots, one dash, b dots, one dash, c dots, one dash, d dots, one dash, and e dots. (Notice that if $b = 0$, we will write two dashes in a row.) Conversely, given a sequence of 20 dots and 4 dashes, we can recover a, b, c, d, and e by counting the number of dots before the first dash, the number of dots between the first dash and the second dash, and so forth. Thus we have reduced the problem to counting the number of sequences of 20 dots and 4 dashes. This is the same as choosing the location of the 4 dashes among $20 + 4$ possible locations, so the total number of these sequences is $\binom{24}{4} = \frac{24 \cdot 23 \cdot 22 \cdot 21}{4 \cdot 3 \cdot 2 \cdot 1} = 23 \cdot 22 \cdot 21 = \boxed{10626}$.

9. (*Ying-Ying Tran*) Is 23 a square mod 41? Is 15 a square mod 41?

Solution: Let p and q be two odd primes. The law of quadratic reciprocity states that if $q \equiv 1 \pmod 4$ then p is a square mod q if and only if q is a square mod p, and if $q \equiv 3 \pmod 4$ then p is a square mod q if and only if $-q$ is a square mod p. Thus 23 is a square mod 41 if and only if 41 is a square mod 23. Notice that $41 \equiv 18 = 2 \cdot 3^2 \pmod{23}$. Thus 41 is a square mod 23 if and only if 2 is a square mod 23. It is known that 2 is a square mod p if and only if $p \equiv \pm 1 \pmod 8$. Since $23 \equiv -1 \pmod 8$, we conclude that 2 is a square mod 23, so $\boxed{\text{23 is a square mod 45}}$ by the above argument.

To determine if 15 is a square mod 41, we first determine if its prime divisors 3 and 5 are squares mod 41. Again, since $41 \equiv 1 \pmod 4$, we know that 3 is a square mod 41 if and only if 41 is a square mod 3. Notice that $41 \equiv 2 \pmod 3$, which is not a square mod 3, so 3 is not a square mod 41. Likewise, 5 is a square mod 41 if and only if 41 is a square mod 5. Since $41 \equiv 1 \pmod 5$, we know that it is a square mod 5, so 5 is a square mod 41. Since 15 is the product of a square and a non-square mod 41, we know that $\boxed{\text{15 is not a square mod 41}}$.

This problem can also be solved without the law of quadratic reciprocity by computing the powers of 2 modulo 41. We compute that the powers of 2 are:

$$1 \quad 2 \quad 4 \quad 8 \quad 16 \quad -9 \quad -18 \quad 5 \quad 10 \quad 20 \quad -1 \quad -2 \quad -4 \quad -8 \quad -16 \quad 9 \quad 19 \quad -5 \quad -10 \quad -20 \quad 1$$

Note that there are 20 powers of 2, which is exactly half the number of nonzero residues mod 41. There are several ways of showing that this means that the powers of 2 are precisely the squares mod 41, so we can check whether or not a number is a square mod 41 by checking whether or not it is in the above list. One way to do this is to notice that $7^2 \equiv 8 = 2^3 \pmod{41}$. We the note that $2 = 1 \cdot 2 \equiv 2^{20} \cdot 2 = 2^{21} = (2^3)^7 \equiv (7^2)^7 = (7^7)^2 \pmod{41}$. Thus 2 is a square mod 41, so any power of 2 is also a square mod 41. Moreover, since $n^2 = (-n)^2$, there are at most 20 distinct squares mod 41. Since there are 20 powers of 2, a number that is not a power of 2 cannot be a square. We see that $23 \equiv 18$ is in this list, whereas 15 is not. Thus $\boxed{\text{23 is a square mod 41, whereas 15 is not}}$.

10. (*Ying-Ying Tran*) Let $\phi(n)$ be the number of positive integers less than or equal to n that are relatively prime to n. Compute $\sum_{d|15015} \phi(d)$.

Solution: Computing $\sum_{d|n} \phi(d)$ for small values of n can lead one to guess that $\sum_{d|n} \phi(d) = n$, leading to an answer of $\boxed{15015}$. One can prove this equation by writing the fractions $\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n}$, and then rewriting them in lowest terms. Note that the denominator becomes a divisor of n, and the numerator is relative prime to the denominator and less than or equal to the denominator. One can check that if k is relatively prime to d and less than or equal to d, where $d|n$, then the fraction $\frac{k}{d}$ appears exactly once in this list. Thus the pairs (k, d) where $d|n$ and k is relatively prime to d and less than or equal to d correspond to the n fractions $\frac{1}{n}, \ldots, \frac{n}{n}$. We conclude that $\sum_{d|n} \phi(d) = n$, as desired.

11. (*Yasha Berchenko-Kogan* and *Ryan Muller*) Compute the largest possible volume of an regular tetrahedron contained in a cube with volume 1.

Solution: Let the vertices of the cube be the eight points in space with coordinates either 0 or 1. The largest possible regular tetrahedron contained in a cube has vertices at "every other" vertex of the cube, for example at $(0,0,0)$, $(0,1,1)$, $(1,0,1)$, and $(1,1,0)$. In fact, this is the largest tetrahedron in the cube even if we don't require the tetrahedron to be regular. Its side length is $\sqrt{0^2+1^2+1^2} = \sqrt{2}$, so we can use the formula for the area of an equilateral triangle to compute that the area of the base of the tetrahedron is $\frac{(\sqrt{2})^2\sqrt{3}}{4} = \frac{\sqrt{3}}{2}$. To compute the height of the tetrahedron, we compute that the centroid of the triangle with vertices $(0,1,1)$, $(1,0,1)$, and $(1,1,0)$ is $\frac{1}{3}((0,1,1)+(1,0,1)+(1,1,0)) = \left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right)$. Thus the height is the distance from $(0,0,0)$ to $\left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right)$, which is $\frac{2\sqrt{3}}{3}$. Using the formula for the volume of a tetrahedron, we see that the answer is $\frac{1}{3}\frac{\sqrt{3}}{2}\frac{2\sqrt{3}}{3} = \boxed{\frac{1}{3}}$.

There are several ways to see that this tetrahedron is the largest. Let p_1, p_2, p_3, and p_4 denote the vertices of the tetrahedron and $V(p_1, p_2, p_3, p_4)$ denote its volume. We would like to maximize V subject to the constraint that p_1, p_2, p_3, and p_4 must lie inside the cube. Notice that V is a linear function in each coordinate. (If we move one vertex of the tetrahedron while leaving the other three fixed, the volume of the tetrahedron will be a linear function in terms of the coordinates of that vertex.) A basic result of linear programming is that the maximum of a linear function over a region defined by linear equations (such as a cube) occurs at a vertex of the region. We conclude that p_1, p_2, p_3, and p_4 must all be vertices of the cube, and we can find the largest value of V by comparing all of the choices of four vertices of the cube.

12. (*Sam Elder*) Compute the number of ways to cover a 4×4 grid with dominoes.

Solution: Consider the middle four squares. If two dominoes cover these, there are two ways to cover the rest. Since there are two ways for two dominoes to cover the middle two squares, this makes 4 ways in all. If three dominoes cover them, there is exactly one way to cover the rest of the 4x4 square (insert diagram). Since there are four possible pairs to be covered by a single domino, and two possible choices for each of the other two dominoes, this makes 16 possible ways in all. Finally, if there are four different dominoes intersecting the middle square, there are two ways each could be placed, and then the other domino in that quadrant is determined, so there are 16 ways in this case too. In all this makes $4 + 16 + 16 = \boxed{36}$.

13. *(Connor Ahlbach)* A collection of points is called *mutually equidistant* if the distance between any two of them is the same. For example, three mutually equidistant points form an equilateral triangle in the plane, and four mutually equidistant points form a regular tetrahedron in three-dimensional space. Let A, B, C, D, and E be five mutually equidistant points in four-dimensional space. Let P be a point such that $AP = BP = CP = DP = EP = 1$. Compute the side length AB.

Solution: Curiously, the easiest way to do this problem involves going all the way up to five dimensions. There, it's easy to write down five mutually equidistant points in coordinates: Let $A = (x, 0, 0, 0, 0)$, $B = (0, x, 0, 0, 0)$, $C = (0, 0, x, 0, 0)$, $D = (0, 0, 0, x, 0)$, and $E = (0, 0, 0, 0, x)$. Using the distance formula, we can check that the distance between any two of these points is $\sqrt{x^2 + (-x)^2 + 0^2 + 0^2 + 0^2} = x\sqrt{2}$. Since P is the center of the simplex $ABCDE$, its coordinates are simply the average of the coordinates of the five points A, B, C, D, and E. Thus P has coordinates $\left(\frac{x}{5}, \frac{x}{5}, \frac{x}{5}, \frac{x}{5}, \frac{x}{5}\right)$. Again, we use the distance formula to compute

$$1 = AP = \sqrt{\left(\tfrac{4}{5}x\right)^2 + \left(\tfrac{1}{5}x\right)^2 + \left(\tfrac{1}{5}x\right)^2 + \left(\tfrac{1}{5}x\right)^2 + \left(\tfrac{1}{5}x\right)^2} = \frac{2\sqrt{5}}{5}x$$

Thus $x = \frac{5}{2\sqrt{5}} = \frac{\sqrt{5}}{2}$. We conclude that $AB = x\sqrt{2} = \boxed{\dfrac{\sqrt{10}}{2}}$.

14. (*Yasha Berchenko-Kogan*) Ten turtles live in a pond shaped like a 10-gon. Because it's a sunny day, all the turtles are sitting in the sun, one at each vertex of the pond. David decides he wants to scare all the turtles back into the pond. When he startles a turtle, it dives into the pond. Moreover, any turtles on the two neighbouring vertices also dive into the pond. However, if the vertex opposite the startled turtle is empty, then a turtle crawls out of the pond and sits at that vertex. Compute the minimum number of times David needs to startle a turtle so that, by the end, all but one of the turtles are in the pond.

Solution: Number the turtles 0 through 9, in order. David can get all but one turtle into the pond by startling five turtles, as follows, where a * denotes a turtle outside the pond.

Startle	0	1	2	3	4	5	6	7	8	9
	*	*	*	*	*	*	*	*	*	*
0			*	*	*	*	*	*	*	
4			*				*	*	*	*
8			*	*			*			
2							*	*		
6		*								

It remains to show that David can't achieve this by startling just four turtles. There are several ways of doing this, and we will demonstrate one such way. Let's assume for contradiction that he can. If the last turtle that David startles is turtle 0, then right before he startles it, the only turtles that might be outside the pond are turtles, 9, 0, 1, and 5. (Otherwise, there would be more than one turtle outside the pond after he startles turtle 0.) We know that after David startles the third turtle, a turtle must appear on the opposite vertex or already be there. Since the only turtles outside the pond after turn three are 9, 0, 1, and 5, we know that the third turtle that David startled had to be 4, 5, 6, or 0. However, the third turtle that David startled could not have been turtle 0, because then he wouldn't have been able to startle it on his fourth turn. Since the third startled turtle is 4, 5, or 6, we know that turtle 5 is in the pond after David's third turn. We conclude that on David's last turn, the vertex opposite the startled turtle must be empty. Thus, any turtle outside the pond after one of the first three turns must jump back in on a later turn. (We can't just have a turtle sitting outside the pond while the other nine turtles jump in.)

We will henceforth no longer assume that the last turtle that David startles is turtle 0. Assume for contradiction that there are three turtles in a row that are never startled. Then the middle turtle never jumps into the pond, which is a contradiction. Thus for every three adjacent turtles, David must startle one of them on some turn.

Now assume for contradiction that there are five turtles in a row such that only one of them is ever startled. Assume without loss of generality that these are turtles 8, 9, 0, 1, and 2. Since David must startle turtle 8, 9, or 0, and he must startle turtle 0, 1, or 2. Since he only startles one turtle between 8 and 2, he must startle turtle 0. Since he doesn't startle turtles 8 or 9, he must startle turtle 7. Likewise, he must startle turtle 3. We can assume without loss of generality that David startles turtle 7 before he startles turtle 3. After he startles turtle 3, we know that turtle 8 must be outside the pond. In order to get it back in, David must startle turtle 7, 8, or 9. We know that David never startles turtles 8 or 9. Moreover, David can't startle turtle 7 a second time, because then he would use up his four turns on turtles 0, 3, and 7, and so turtle 5 would never jump into the pond. We have reached a contradiction, so we know that for every five turtles in a row, David must startle at least two of them.

To summarize, David must startle at least one turtle in every group of three adjacent turtles, and he must startle at least two turtles in every group of five adjacent turtles. One can check that this means that David must startle turtles 1, 4, 6, and 9 in some order (up to rotating the vertex labels). Like earlier, we can assume without loss of generality that David startles turtle 1 before he startles turtle 6. After he startles turtle 6, turtle 1 crawls back out of the pond. Since David doesn't startle turtles 0 or 2 and David doesn't startle turtle 1 a second time, turtle 1 never jumps back into the pond, a contradiction.

Thus David needs $\boxed{5}$ turns so that all but one of the turtles are in the pond.

15. (*Yasha Berchenko-Kogan*) The game *hexapawn* is played on a 3×3 chessboard. Each player starts with three pawns on the row nearest him or her. The players take turns moving their pawns. Like in chess, on a player's turn he or she can either

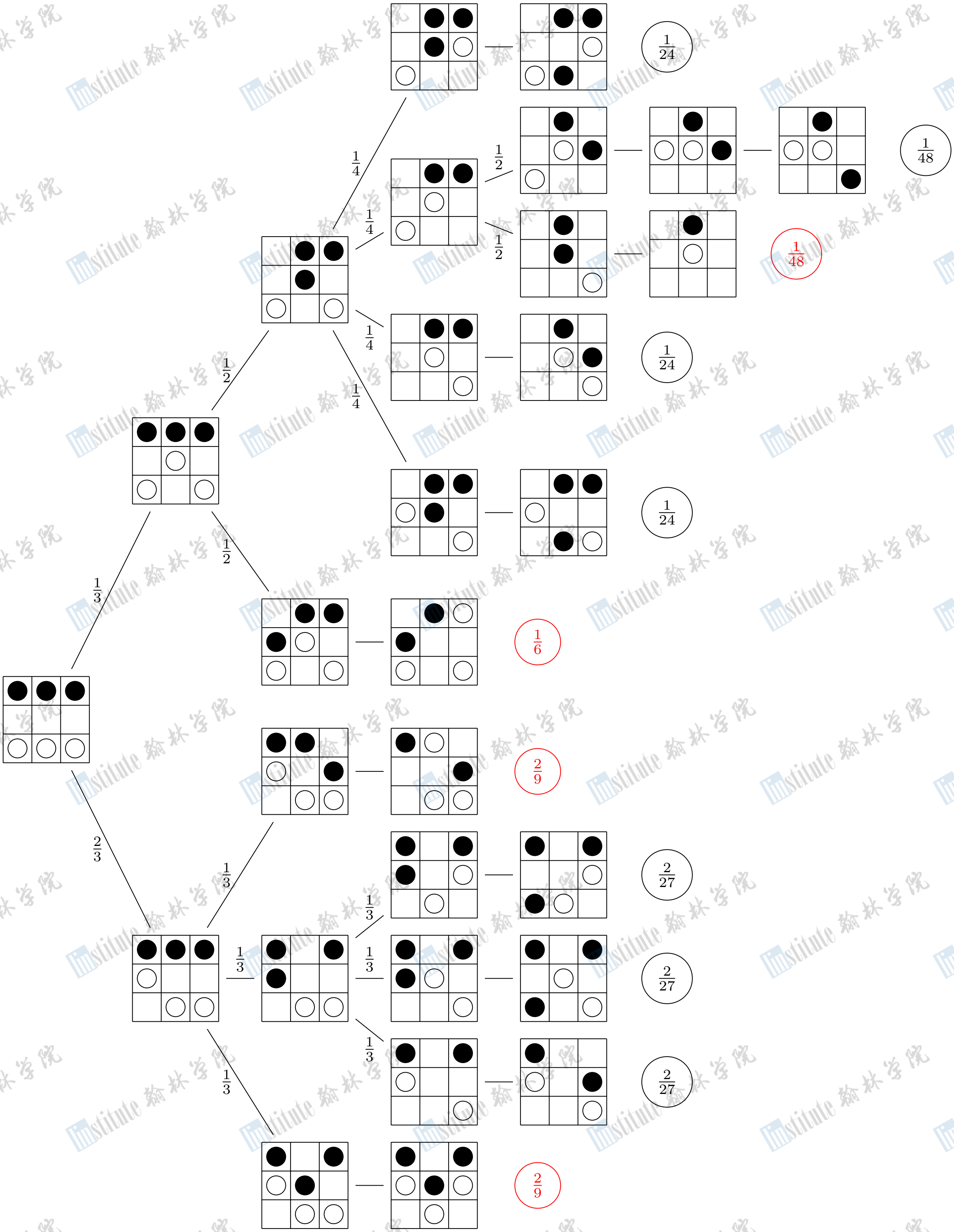
- move a pawn forward one space if that square is empty, or
- capture an opponent's pawn by moving his or her own pawn diagonally forward one space into the opponent's pawn's square.

A player wins when either

- he or she moves a pawn into the last row, or
- his or her opponent has no legal moves.

Eve and Fred are going to play hexapawn. However, they're not very good at it. Each turn, they will pick a legal move at random with equal probability, with one exception: If some move will immediately win the game (by either of the two winning conditions), then he or she will make that move, even if other moves are available. If Eve moves first, compute the probability that she will win.

Solution: We construct a tree of possibilities on the same page, using symmetry to decrease the amount of computation. The numbers in red are the probabilities of that particular winning configuration for Eve. Thus the probability that Eve wins is the sum of the red numbers, which is $\frac{1}{48} + \frac{1}{6} + \frac{2}{9} + \frac{2}{9} = \boxed{\dfrac{91}{144}}$.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