Caltech Harvey Mudd Mathematics Competition

Individual Round Solutions **February 20, 2010**

1. (*Sam Elder*) Compute the degree of the least common multiple of the polynomials $x-1,\ x^2-1,\ x^3-1, \ldots,\ x^{10}-1$.

   Solution 1: The roots of x^{n-1} are the nth roots, of unity. It's clear that none of these roots are repeated, so the gcd will be the polynomial with roots at each of the 1st through 10th roots of unity. Since these are just $e^{\frac{i2\pi m}{n}}$ for $0 \leq m < n$ and $(m,n)=1$, and $n \leq 10$, we must count the number of reduced fractions between 0 and 1 whose denominators are at most 10. There is 1 with denominator 1 (1/1), 1 with 2 (1/2), 2 with 3 (1/3 and 2/3), 2 with 4, 4 with 5, 2 with 6, 6 with 7, 4 with 8, 6 with 9, and 4 with 10. (In general, there are $\phi(n)$ with n, where ϕ is Euler's totient function.) Adding these up yields $\boxed{32}$.

   Solution 2: We can factor these as products of irreducible (cyclotomic) polynomials: $x-1, (x-1)(x+1), (x-1)(x^2+x+1), (x-1)(x+1)(x^2+1), (x-1)(x^4+x^3+x^2+x+1), (x-1)(x+1)(x^2+x+1)(x^2-x+1), (x-1)(x^6+x^5+x^4+x^3+x^2+x+1), (x-1)(x+1)(x^2+1)(x^4+1), (x-1)(x^2+x+1)(x^6+x^3+1)$, and $(x-1)(x+1)(x^4+x^3+x^2+x+1)(x^4-x^3+x^2-x+1)$, and then combine all factors that show up. Their degrees are $1+1+2+2+4+2+6+4+6+4 = \boxed{32}$ again. ($\phi(n)$ is also the degree of the nth cyclotomic polynomial.)

2. (*Yasha Berchenko-Kogan*) A line in the xy plane is called *wholesome* if its equation is $y = mx+b$ where m is rational and b is an integer. Given a point with integer coordinates (x,y) on a wholesome line l, let r be the remainder when x is divided by 7, and let s be the remainder when y is divided by 7. The pair (r,s) is called an *ingredient* of the line l. The (unordered) set of all possible ingredients of a wholesome line l is called the *recipe* of l. Compute the number of possible recipes of wholesome lines.

   Solution: Since we're working modulo 7, a natural collection of lines to consider is $y = mx+b$ where $m \in \{0,1,2,3,4,5,6\}$ and $b \in \{0,1,2,3,4,5,6\}$. The recipe of one of these lines contains $(0,b)$ and $(1, \overline{m+b})$, where $\overline{m+b}$ denotes the remainder when $m+b$ is divided by 7. It is easy to check that this recipe has no other ingredients with first coordinate 0 or 1. We conclude that these 49 lines all have different recipes.

   We need to check that there are no other possible recipes. Let $y = mx+b$ be any wholesome line, and let $m = \frac{c}{d}$, where c and d are relatively prime integers. Thus we have $dy = cx+db$. As long as s is not divisible by 7, we can find a number t such that $td \equiv 1 \pmod 7$. We conclude that $tdy \equiv tcx + tdb \pmod 7$, so $y \equiv tcx + b \pmod 7$. Thus the line $y = mx+b$ has the same recipe as the line $y = \overline{tc}x + b$, so it is one of the 49 lines above.

   However, if d is divisible by 7, then $cx = d(y-b)$ is also divisible by 7. Since c and d are relatively prime, we know that c is not divisible by 7, so x must be divisible by 7. Thus the first coordinate of any ingredient of this line is 0. We note that the line $y = \frac{1}{7}x$ has the recipe $\{(0,0),(0,1),(0,2),(0,3),(0,4),(0,5),(0,6)\}$. Are there any other recipes of lines where d is divisible by 7? Notice that for any integer n, $(x,y) = (d, nc+b)$ is an integer point on the line $y = mx+b$, with corresponding ingredient $(0, \overline{nc+b})$. Since c is not divisible by 7, we know that $\{\overline{nc+b} \mid n \in \mathbb{Z}\} = \{0,1,2,3,4,5,6\}$. The recipe contains all ingredients with first coordinate 0, and so it is the same recipe as for the line $y = \frac{1}{7}x$.

   Therefore, we have a total of $49 + 1 = \boxed{50}$ recipes.

3. (*Ying-Ying Tran*) Let $\tau(n)$ be the number of distinct positive divisors of n. Compute $\sum_{d|15015} \tau(d)$, that is, the sum of $\tau(d)$ for all d such that d divides 15015.

   Solution: A *multiplicative function* is a function f defined on the positive integers such that $f(ab) = f(a)f(b)$ whenever a and b are relatively prime. It is easy to use the formula for the number of positive divisors of n to check that τ is indeed a multiplicative function.

   Moreover, it is not difficult to prove that if f is a multiplicative function, then $g(n) = \sum_{d|n} f(d)$ is also a multiplicative function. In particular, $\sum_{d|n} \tau(d)$ is multiplicative. Notice that $15015 = 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$.

Notice also that for any prime p, we have $\sum_{d|p} \tau(p) = \tau(1) + \tau(p) = 1 + 2 = 3$. Therefore,

$$\sum_{d|15015} \tau(d) = \left(\sum_{d|3} \tau(d)\right)\left(\sum_{d|5} \tau(d)\right)\left(\sum_{d|7} \tau(d)\right)\left(\sum_{d|11} \tau(d)\right)\left(\sum_{d|13} \tau(d)\right) = 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 = \boxed{243}$$

4. (*Yasha Berchenko-Kogan*) Suppose $2202010_b - 2202010_3 = 71813265_{10}$. Compute b. (n_b denotes the number n written in base b.)

   Solution: A quick computation yields $2202010_b = 71813265 + 2(3)^6 + 2(3)^5 + 2(3)^3 + 3 = 71\,815\,266$. To get a rough bound on b, we see that $2000000_b = 2 \cdot b^6 < 71\,815\,266 < 3 \cdot b^6 = 3000000_b$. Since $2 \cdot 20^6 = 128\,000\,000 > 71\,815\,266$, we know that $b < 20$. Since $2 \cdot 10^6 = 2\,000\,000 < 71\,815\,266$, we know that $b > 10$. Notice that b divides $2202010_b = 71\,815\,266$. Factoring, we see that $71\,815\,266 = 18 \cdot 3\,989\,737$. We can check that no prime under 20 divides $3\,989\,737$, so the only number between 10 and 20 that divides 2202010_b is 18. We conclude that $\boxed{b = 18}$.

   One can also solve this problem by letting $d = 20 - b$ and approximating $71813265 \approx 2 \cdot (20 - d)^6 \approx 2(64 \cdot 10^6 - 6 \cdot 32 \cdot 10^5 d)$. Thus $d \approx \frac{128\,000\,000 - 71\,813\,265}{38\,400\,000} \approx 1.5$, suggesting that b is either 18 or 19. One could then refine the approximation to guess that the answer is 18, or use the first method to check that 2202010_b is divisible by 18 but not 19.

5. (*Connor Ahlbach*) Let $x = (3 - \sqrt{5})/2$. Compute the exact value of $x^8 + 1/x^8$.

   Solution: Note that $(3 - \sqrt{5})(3 + \sqrt{5}) = 9 - 5 = 4$. Thus $x \cdot \frac{3+\sqrt{5}}{2} = 1$, so $x + \frac{1}{x} = \frac{3-\sqrt{5}}{2} + \frac{3+\sqrt{5}}{2} = 3$. Squaring both sides, we obtain the equation $x^2 + 2 + x^{-2} = 9$, so $x^2 + x^{-2} = 7$. Squaring both sides again, we find that $x^4 + 2 + x^{-4} = 49$, so $x^4 + x^{-4} = 47$. Squaring both sides one last time, we find that $x^8 + 2 + x^{-8} = 2209$, so $x^8 + \frac{1}{x^8} = \boxed{2207}$.

6. (*Yasha Berchenko-Kogan*) Compute the largest integer that has the same number of digits when written in base 5 and when written in base 7. Express your answer in base 10.

   Solution: Let n be the largest such number, and assume that n has k digits in base 5 and in base 7. Then $5^{k-1} \le n \le 5^k - 1$ and $7^{k-1} \le n \le 7^k - 1$. In particular, we see that $7^{k-1} \le 5^k - 1$. We would like to find the largest k for which this inequality is true. We see that $7^1 = 7 < 24 = 5^2 - 1$, $7^2 = 49 < 124 = 5^3 - 1$, $7^3 = 343 < 624 = 5^4 - 1$, and $7^4 = 2401 < 3124 = 5^5 - 1$, but $7^5 = 16807 > 15624 = 5^6 - 1$. Thus the largest possible value of k is 5. Therefore, the answer is the largest 5-digit number in base 5, which is $5^5 - 1 = \boxed{3124} = 44444_5 = 12052_7$.

7. (*Sam Elder*) Three circles with integer radii a, b, c are mutually externally tangent, with $a \le b \le c$ and $a < 10$. The centers of the three circles form a right triangle. Compute the number of possible ordered triples (a, b, c).

   Solution: Using the Pythagorean theorem, we see that $(a + b)^2 + (a + c)^2 = (b + c)^2$. Expanding, we see that $2a^2 + b^2 + c^2 + 2ab + 2ac = b^2 + c^2 + 2bc$. We conclude that $a^2 + ab + ac = bc$, so $(b - a)(c - a) = a^2 - ab - ac + bc = 2a^2$. Notice that $b \ge a$ and $c \ge a$, so $b - a$ and $c - a$ are positive divisors of $2a^2$. Thus for each particular value of a, we get a solution (b, c) for each pair of positive numbers whose product is $2a^2$. Since we have the condition that $b \le c$, this number is equal to half the number of divisors of $2a^2$. We compute this number for $1 \le a \le 9$:

a	1	2	3	4	5	6	7	8	9
$2a^2$	2	8	18	32	50	72	98	128	162
$\frac{1}{2}\tau(2a^2)$	1	2	3	3	3	6	3	4	5

   Thus, the total number of solutions (a, b, c) with $a \le b \le c$ is $1 + 2 + 3 + 3 + 3 + 6 + 3 + 4 + 5 = \boxed{30}$.

8. (*Yasha Berchenko-Kogan*) Six friends are playing informal games of soccer. For each game, they split themselves up into two teams of three. They want to arrange the teams so that, at the end of the day,

each pair of players has played at least one game on the same team. Compute the smallest number of games they need to play in order to achieve this.

Solution: Of the three players on the same team in the first game, two of them must also be on the same team in the second game. Call these two players A and B, and call the third player on their team in the first game C. Let D and E be C's teammates in the second game, and let F be the sixth player. Thus, in the first game, A, B, and C played against D, E, and F. In the second game, A, B, and F played against C, D, E. After the first two games, A and D haven't yet played with each other, B and E haven't yet played with each other, and C and F haven't yet played with each other. It is impossible to set up the third game so that A plays with D, B plays with E, and C plays with F. Thus the six friends will not be able to achieve their goal after three games. However, they can achieve their goal in four games. For example, if A, B, and D can play against C, E, and F in the third game, and A, B, and E can play against C, D, and F in the fourth game. Thus the answer is $\boxed{4}$.

9. (*Yasha Berchenko-Kogan*) Let A and B be points in the plane such that $AB = 30$. A circle with integer radius passes through A and B. A point C is constructed on the circle such that $\overline{AC}$ is a diameter of the circle. Compute all possible radii of the circle such that BC is a positive integer.

Solution: Since $\overline{AC}$ is the diameter of the circle and B is on the circle, we know that $\angle ABC$ is a right angle. Let r be the radius of the circle, and let $x = BC$. By the Pythagorean theorem, $30^2 + x^2 = (2r)^2$. Since 30^2 is even and $(2r)^2$ is even, we conclude that x must be even. Let $y = \frac{x}{2}$. Then $225 = 15^2 = r^2 - y^2 = (r+y)(r-y)$. Since $r+y$ and $r-y$ are integers, they are factors of 225. The factors of 225 are $1, 3, 5, 9, 15, 25, 45, 75, 225$. If $r+y = 225$ and $r-y = 1$, then $r = \frac{1}{2}(225+1) = 113$. If $r+y = 75$ and $r-y = 3$, then $r = 39$. If $r+y = 45$ and $r-y = 5$, then $r = 25$. If $r+y = 25$ and $r-y = 9$, then $r = 17$. We can't have $r+y = 15$ and $r-y = 15$ because then y would be zero. Thus the possible radii are $\boxed{17, 25, 39, 113}$.

10. (*Tim Black*) Each square of a 3×3 grid can be colored black or white. Two colorings are the same if you can rotate or reflect one to get the other. Compute the total number of unique colorings.

Solution: Unfortunately, you can't just take the number of ways to color the grid, 512, and divide by 8 to account for the seven other equivalent colorings you can get by reflecting and rotating, because some colorings, such as the coloring where all squares are black, have no colorings that are equivalent to them. The easiest way to do this problem is with Burnside's lemma, which tells you how to count things up to transformations such as rotations and reflections. To use Burnside's lemma, we need to do the following:

- Describe the objects without taking the transformations into account.
  In this problem, the objects are colorings of a 3×3 grid.
- Describe all of the transformations that we're allowed to perform on the objects.
  In this problem, we can rotate and reflect the grid. There are four possible rotations: 90°, 180°, 270°, and 0° (leaving the grid in place). There are also four ways to reflect the grid: across the vertical axis, across the horizontal axis, and across one of the two diagonal axes.
- For each transformation, count how many objects remain the same after the transformation is applied. In other words, count how many objects are symmetric with respect to each transformation.
  The colorings that are symmetric with respect to a 90° or a 270° rotation have the following form, where A, B, and C denote arbitrary colors. There are $2^3 = 8$ of them.

B	C	B
C	A	C
B	C	B

  The colorings that are symmetric with respect to a 180° rotation have the following form, where A, B, C, D, and E denote arbitrary colors. There are $2^5 = 32$ of them.

B	C	D
E	A	E
D	C	B

Any coloring remains the same after a $0°$ rotation, so there are $2^9 = 512$ of them.

The colorings that are symmetric with respect to reflections across the vertical axis have the following form, where A, B, C, D, E, and F denote arbitrary colors. There are $2^6 = 64$ of them.

A	B	A
C	D	C
E	F	E

It is easy to check that for each of the other three reflections, there are also $2^6 = 64$ colorings that are symmetric with respect to it.

- The number of objects up to transformation is the average of the numbers of objects that are symmetric with respect to a particular transformation.

  In this case, the number of colorings symmetric with respect to $90°$ and $270°$ rotations is 8, the number of colorings symmetric with respect to $180°$ rotations is 32, the number of colorings symmetric with respect to $0°$ rotations is 512, and the number of colorings symmetric with respect to each of the four reflections is 64. Thus the total number of colorings up to rations and reflections is $\frac{1}{8}(8 + 8 + 32 + 512 + 64 + 64 + 64 + 64) = 1 + 1 + 4 + 64 + 8 + 8 + 8 + 8 = \boxed{102}$.

There are several ways of doing this problem without Burnside's lemma. One way is to notice that exactly half of the colorings will have a white center square, and exactly half of the colorings will have a black center square. Thus it's enough to count the number of colorings of the eight outside squares, and then multiply the answer by two. We can do this with a careful case analysis.

Assume all four of the corners are black. Either all four sides are black, three of them are black, two adjacent ones are be black, two opposite ones are black, one of them is black, or all of them are white. We have a total of 6 colorings.

Now assume three of the corners are black. There is one possible coloring if all four sides are black. If three of the sides are black, then the white side can be adjacent to the white corner or not, so there are two colorings. We can have two adjacent black sides in three distinct ways: they might surround the white corner, be adjacent to the white corner, or be opposite the white corner. If we have two black sides that are opposite each other, then one can check that there is just one possible coloring. If three of the sides are white, then we again have two colorings, and if all four sides are white, we again have one colorings. Thus there are 10 possible colorings if three of the corners are black.

Now assume two opposite corners are black. Again, there is one possible coloring if all four sides are black. If three of the sides are black, one can check that there is still only one possible coloring up to rotations and reflections. If two adjacent sides are black, then they either surround a black corner or a white corner, so we have two possibilities. If two opposite sides are black, then again there is only one possible coloring. If three of the sides are white, then there is again just one possible coloring, and similarly for the case where all four sides are white. Thus there are 7 possible colorings if two opposite corners are black.

Finally, assume two adjacent corners are black. There is one possible coloring if all four sides are black. There are three possibilities if three of the sides are black: the white side can be between the black corners, adjacent to one of the black corners, or opposite the black corners. If two opposite sides are black, then we have two distinct colorings: the side inbetween the black corners can be either black or white. If two adjacent sides are black, then there are again two possibilities: the black sides can surround a black corner or a white corner. If three of the sides are white, there are again three colorings, and if all four sides are white, there is again just one coloring. Thus there are 12 possible colorings if two adjacent corners are black.

The number of colorings with three white corners is the same as the number of colorings with three black corners, which is 10. The number of colorings with all white corners is the same as the number of colorings with all black corners, which is 6.

Thus there are $6 + 10 + 7 + 12 + 10 + 6 = 51$ ways to color the outside eight squares, so there are $2 \cdot 51 = \boxed{102}$ colorings of the entire 3×3 grid.

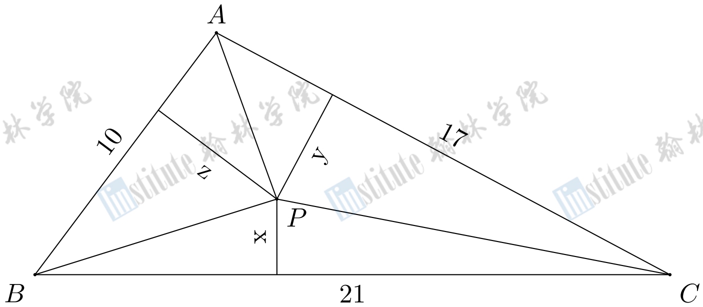
11. (*Yasha Berchenko-Kogan*) Compute all positive integers n such that the sum of all positive integers that are less than n and relatively prime to n is equal to $2n$.

Solution: Clearly, $n = 1$ and $n = 2$ do not satisfy the condition of the problem. Assume henceforth that $n > 2$. Notice that if k is a positive number relatively prime to n, then $n - k$ is also relatively prime to n. Since $n > 1$, we know that $n - k$ is positive. Since $n \neq 2$, we know that $n \neq n - k$. (Otherwise, $\frac{n}{2}$ would be relatively prime to n, which is impossible from $n \neq 2$.) Thus we can split the numbers relatively prime to n into pairs $(k, n - k)$ whose sum is n. We conclude that the sum of all positive numbers less than or equal to n and relatively prime to n is $\frac{n}{2}\phi(n)$, where $\phi(n)$ is the number of positive numbers less than or equal to n that are relatively prime to n. The problem tells us that this sum is $2n$, so $\phi(n) = 4$.

We can decompose n as a product of primes, so $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$. It is a standard fact that $\phi(n) = p_1^{\alpha_1 - 1}(p_1 - 1)p_2^{\alpha_2 - 1}(p_2 - 1) \cdots p_s^{\alpha_s - 1}(p_s - 1)$. Since $\phi(n) = 4$, we see that $p_i - 1 \leq 4$, so $p_i \leq 5$. Thus the only primes that can divide n are 2, 3, and 5. Moreover, 3 and 5 can't both divide n, because then $\phi(n)$ would be divisible by $(3 - 1)(5 - 1) = 8$. We also know that 5^2 can't divide n, because then $\phi(n)$ would be divisible by $5(5 - 1) = 20$. Likewise 3^2 can't divide n because $\phi(n)$ cannot be divisible by $3(3 - 1) = 6$. Finally, 2^4 can't divide n because then $\phi(n)$ would be divisible by $2^3(2 - 1) = 8$. Thus the only numbers that could have $\phi(n) = 4$ are $2^2, 2^3, 3, 2 \cdot 3, 2^2 \cdot 3, 2^3 \cdot 3, 5, 2 \cdot 5, 2^2 \cdot 5,$ and $2^3 \cdot 5$. We can quickly compute $\phi(n)$ for all of these numbers, and we see that the only ones that have $\phi(n) = 4$ are $2^3, 2^2 \cdot 3, 5,$ and $2 \cdot 5$. Thus the positive integers satisfying the condition in the problem are $\boxed{5, 8, 10, \text{ and } 12}$.

12. (*Yasha Berchenko-Kogan*) The distance between a point and a line is defined to be the smallest possible distance between the point and any point on the line. Triangle ABC has $AB = 10$, $BC = 21$, and $CA = 17$. Let P be a point inside the triangle. Let x be the distance between P and $\overleftrightarrow{BC}$, let y be the distance between P and $\overleftrightarrow{CA}$, and let z be the distance between P and $\overleftrightarrow{AB}$. Compute the largest possible value of the product xyz.

Solution: We first draw a diagram.



Let K be the area of $\triangle ABC$. The segments $\overline{PA}$, $\overline{PB}$, and $\overline{PC}$ split $\triangle ABC$ into three triangles. The three triangles have area $\frac{21x}{2}$, $\frac{17y}{2}$, and $\frac{10z}{2}$, so $K = \frac{1}{2}(21x + 17y + 10z)$. Using Heron's formula, we find that the area of the triangle is $K = \sqrt{24 \cdot 3 \cdot 7 \cdot 14} = 84$. Thus $21x + 17y + 10z = 168$. It is not hard to check that the converse is also true: If x, y, and z are positive numbers satisfying $21x + 17y + 10z$, then there exists a point P such that x, y, and z are the distances to the edges of the triangle. Using the fact that the geometric mean is less than or equal to the arithmetic mean, we see that $56 = \frac{21x+17y+10z}{3} \geq \sqrt[3]{(21x)(17y)(10z)}$. We conclude that $xyz \leq \frac{56^3}{21 \cdot 17 \cdot 10} = \frac{56^2 \cdot 4}{3 \cdot 17 \cdot 5} = \frac{12544}{255}$ and that equality occurs when $21x = 17y = 10z$. Thus if $x = \frac{8}{3}$, $y = \frac{56}{17}$, and $z = \frac{28}{5}$, then xyz attains its maximum value of $\boxed{\dfrac{12544}{255}}$.

13. (*Yasha Berchenko-Kogan*) This problem was motivated by a paper by Christopher Tuffley, available at `http://tur-www1.massey.ac.nz/~ctuffley/papers/Keg.pdf`

Alice, Bob, David, and Eve are sitting in a row on a couch and are passing back and forth a bag of chips. Whenever Bob gets the bag of chips, he passes the bag back to the person who gave it to him with probability $\frac{1}{3}$, and he passes it on in the same direction with probability $\frac{2}{3}$. Whenever David gets the bag of chips, he passes the bag back to the person who gave it to him with probability $\frac{1}{4}$, and he passes it on with probability $\frac{3}{4}$. Currently, Alice has the bag of chips, and she is about to pass it to Bob when Cathy sits between Bob and David. Whenever Cathy gets the bag of chips, she passes the bag back to the person who gave it to her with probability p, and passes it on with probability $1-p$. Alice realizes that because Cathy joined them on the couch, the probability that Alice gets the bag of chips back before Eve gets it has doubled. Compute p.

Solution: We begin with the general case. Let X is a person who passes the chips back with probability x and passes it on with probability $1-x$, and let Y be a person who passes the chips back with probability y and on with probability $1-y$. Assume that X and Y are sitting next to each other, and we give the chips to X. X and Y will keep passing the chips back and forth between each other until either X passes them to us or Y passes them on. We would like to compute the probability that we get the chips back from X. With probability x, X will give the chips right back to us. Alternatively, X might pass the chips on to Y, who then might pass them back to X, who then might pass them back to us. This scenario happens with probability $(1-x)y(1-x)$. It's also possible that X and Y pass the chips back and forth twice before giving them back to us. This scenario happens with probability $(1-x)yxy(1-x)$. In general, if X and Y pass the chips back and forth k times before giving them back to us, then X will pass the chips on twice (once to give them to Y at the beginning, and once to give them back to us), Y will pass the chips back to X k times, and X will pass the chips back to Y $k-1$ times. Thus the probability that they pass the chips back and forth k times and then give them back to us is $(1-x)^2x^{k-1}y^k$. The total probability that we get the chips back is

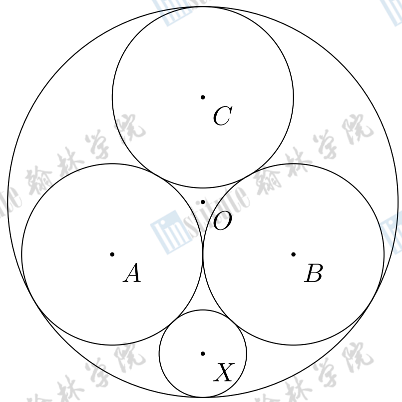
$$x+\sum_{k=1}^{\infty}(1-x)^2x^{k-1}y^k = x+(1-x)^2y\sum_{k=1}^{\infty}(xy)^{k-1} = x+(1-x)^2y\tfrac{1}{1-xy}$$
$$= \frac{(x-x^2y)+(y-2xy+x^2y)}{1-xy} = \frac{x+y-2xy}{1-xy}$$

Thus we can replace X and Y with a single person Z who passes the chips back with probability $\frac{x+y-2xy}{1-xy}$ and on with probability $1-\frac{x+y-2xy}{1-xy}$. The important thing to notice about this formula is that it is symmetric in x and y. This means that we'd get the chips back with the same probability if X and Y were to swap places with each other. In the problem, this means that we'd get the same answer if Cathy sat between David and Eve rather than between Bob and David.

Using this formula, we know that we can replace Bob and David with a single person who passes the chips back with probability $\frac{\frac{1}{3}+\frac{1}{4}-2\frac{1}{3}\frac{1}{4}}{1-\frac{1}{3}\frac{1}{4}} = \frac{\frac{5}{12}}{\frac{11}{12}} = \frac{5}{11}$. Therefore, before Cathy sits down, Alice will get the chips back before Eve gets them with probability $\frac{5}{11}$.

As we mentioned earlier, we will get the same answer if we have Cathy sit between David and Eve. Since we can replace Bob and David with a single person who passes the chips back with probability $\frac{5}{11}$, we see that, after Cathy arrives, the probability that Alice gets the chips back before Eve gets them is $\frac{\frac{5}{11}+p-2\frac{5}{11}p}{1-\frac{5}{11}p} = \frac{5+11p-10p}{11-5p} = \frac{5+p}{11-5p}$. We are told that this probability is double $\frac{5}{11}$, so $\frac{5+p}{11-5p} = \frac{10}{11}$. Thus $55+11p = 110-50p$, so $61p = 55$. Thus $\boxed{p = \dfrac{55}{61}}$.

14. (*Tim Black*) Circle O is in the plane. Circles A, B, and C are congruent, and are each internally tangent to circle O and externally tangent to each other. Circle X is internally tangent to circle O and externally tangent to circles A and B. Circle X has radius 1. Compute the radius of circle O.



Solution: Let R be the radius of circle O, and let r be the radii of circles A, B, and C. Notice that $\triangle AOB$ is a 30°-120°-30° triangle. Thus $AB = \sqrt{3} \cdot OA$. We see that $AB = 2r$. Moreover, if we draw the ray $\overrightarrow{OA}$, we see that $R = OA + r = \frac{\sqrt{3}}{3}AB + r = \frac{2\sqrt{3}+3}{3}r$.

Let M be the midpoint of AB. Since $\triangle AOM$ is a 30°-60°-90° triangle, we see that $OM = \frac{\sqrt{3}}{3}AM = \frac{\sqrt{3}}{3}r$. Since the radius of circle X is 1, we see that $R = OM + MX + 1$. Thus $MX = R - OM - 1 = \left(\frac{3+2\sqrt{3}}{3} - \frac{\sqrt{3}}{3}\right)r - 1 = \frac{3+\sqrt{3}}{3}r - 1$. Since $\triangle AMX$ is a right triangle, we see that $(AM)^2 + (MX)^2 = (AX)^2$. Notice that $AM = r$ and $AX = r + 1$. Thus we have the equations

$$r^2 + \left(\frac{3+\sqrt{3}}{3}r - 1\right)^2 = (r+1)^2$$

$$r^2 + \frac{4+2\sqrt{3}}{3}r^2 - \frac{6+2\sqrt{3}}{3}r + 1 = r^2 + 2r + 1$$

$$\frac{4+2\sqrt{3}}{3}r^2 = \frac{12+2\sqrt{3}}{3}r$$

Thus $r = \frac{12+2\sqrt{3}}{4+2\sqrt{3}} = \frac{6+\sqrt{3}}{2+\sqrt{3}} = (6+\sqrt{3})(2-\sqrt{3}) = 9 - 4\sqrt{3}$, and so $R = \frac{3+2\sqrt{3}}{3}r = \frac{1}{3}(3+2\sqrt{3})(9-4\sqrt{3}) = \boxed{1+2\sqrt{3}}$.

15. (*Yasha Berchenko-Kogan*) Compute the number of primes p less than 100 such that p divides n^2+n+1 for some integer n.

Solution: We would like to find all p such that $n^2+n+1 \equiv 0 \pmod{p}$ for some n. If $n \equiv 1 \pmod{p}$ satisfies $n^2+n+1 \equiv 0 \pmod{p}$, then $3 \equiv 0 \pmod{p}$, so $p = 3$. Assume henceforth that $n \not\equiv 1 \pmod{p}$. Since $n - 1 \not\equiv 0 \pmod{p}$, we can obtain an equivalent equation by multiplying both sides of the equation by $(n-1)$. Thus, for $n \not\equiv 1 \pmod{p}$, we find that $n^2+n+1 \equiv 0 \pmod{p}$ is equivalent to $n^3 - 1 \equiv 0 \pmod{p}$. We have thus reduced the problem to the question of whether there exists an $n \not\equiv 1 \pmod{p}$ such that $n^3 \equiv 1 \pmod{p}$.

Fermat's Little Theorem tells us that if $n \not\equiv 0 \mod p$, then $n^{p-1} \equiv 1 \pmod{p}$. If $p-1$ is not divisible by 3, then either p or $p-2$ is divisible by 3. Assuming $p-1$ is not divisible by 3, if $n^3 \equiv 1 \pmod{p}$, then either $n^p \equiv 1 \pmod{p}$ or $n^{p-2} \equiv 1 \pmod{p}$. We conclude that either $\frac{n^p}{n^{p-1}} \equiv 1 \pmod{p}$ or $\frac{n^{p-1}}{n^{p-2}} \equiv 1 \pmod{p}$. In either case, we conclude that $n \equiv 1 \pmod{p}$, which we assumed was not the case.

Conversely, assume $p-1$ is divisible by 3. Note that if pick $x \not\equiv 0 \pmod{p}$ and set $n = x^{\frac{p-1}{3}}$, then $n^3 = x^{p-1} \equiv 1 \pmod{p}$. However, it might be true that $n \equiv 1 \pmod{p}$. Assume for contradiction that $n = x^{\frac{p-1}{3}} \equiv 1 \pmod{p}$ for all nonzero x modulo p. Then the polynomial $x^{\frac{p-1}{3}} - 1$ has $p-1$ roots modulo p, but its degree is only $\frac{p-1}{3}$, which is a contradiction. Thus there exists some $x \not\equiv 0 \pmod{p}$ such that $n = x^{\frac{p-1}{3}} \not\equiv 1 \pmod{p}$ and $n^3 \equiv 1 \pmod{p}$.

We conclude that the only primes p such that $p \mid n^2 + n + 1$ for some n are $p = 3$ and all p satisfying $3 \mid p-1$. If $3 \mid p-1$, we know that $p-1$ is even, so $6 \mid p-1$. We can quickly check all the numbers less than 100 that are congruent to 1 modulo 6 to see if they are prime. We find that these primes are $7, 13, 19, 31, 37, 43, 61, 67, 73, 79,$ and 97. There are 11 of them. Since $p = 3$ also satisfies the condition of the problem, we conclude that there are $\boxed{12}$ such primes.