Power Round

Caltech Harvey Mudd Mathematics Competition

February 20, 2010

The *roots*, also called *zeroes*, of a function f are the values x such that $f(x) = 0$. You're familiar with the quadratic formula $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$, which computes the roots of a quadratic polynomial ax^2+bx+c in terms of a, b, and c. In this problem you will derive the cubic formula, which computes the roots of a cubic polynomial $ax^3 + bx^2 + cx + d$ with $a \neq 0$.

For this part of the contest, you must fully justify all of your answers unless otherwise specified. In your solutions, you may refer to the answers of earlier problems (but not later problems or later parts of the same problem), even if you were not able to solve those problems.

1. (a) Show that $\sqrt[3]{2} + \sqrt[3]{4}$ is a root of the polynomial $x^3 - 6x - 6$.

   (b) Show that $\sqrt[3]{u} + \sqrt[3]{v}$ is a root of the polynomial $x^3 - (3\sqrt[3]{u}\sqrt[3]{v})x - (u + v)$.

2. (a) Using part 1b, find a real root of $x^3 - 12x - 34$.

   (b) In the complex numbers, $x^3 - 12x - 34$ has three roots. Find the other two roots. (It might help to use the third root of unity $\omega = \frac{-1+i\sqrt{3}}{2}$ when expressing your answers.)

3. In part 1b, you found a root of $x^3 - (3\sqrt[3]{u}\sqrt[3]{v})x - (u + v)$. Find the other two roots of this polynomial in the complex numbers in terms of $\sqrt[3]{u}$ and $\sqrt[3]{v}$.

4. Find all of the roots in the complex numbers of a polynomial of the form $x^3 + cx + d$ in terms of c and d.

5. Let $f(x) = x^3 - 3x^2 + 2x + 3$.

   (a) Make a rough sketch of the graph of f. You do not need to justify your answer to this part.

   (b) Prove the graph of f is symmetric with respect to 180° degree rotations about some point P in the plane. Find the coordinates of P.

   (c) Let x_0 be the x-coordinate of P from part 5b. Show that $f(x + x_0)$ is a cubic polynomial whose x^2 coefficient is zero.

6. Let $f(x) = x^3 + bx^2 + cx + d$.

   (a) Find a number x_0 in terms of b, c, and d, such that $f(x + x_0)$ is a cubic polynomial whose x^2 coefficient is zero.

   (b) In the polynomial $f(x + x_0)$, find the x coefficient and the constant coefficient in terms of b, c, and d.

7. Carefully explain how you would use the answers to the above problems to find all of the roots in the complex numbers of a polynomial of the form $ax^3 + bx^2 + cx + d$, assuming that $a \neq 0$. (You do not need to write down the general cubic formula to obtain full credit on this part.)