Caltech Harvey Mudd Mathematics Competition

In this round, problems will depend on answers to other problems.

Part I

1. Let **F** be the answer to problem number 6. You want to tile a nondegenerate square with side length **F** with 1×2 rectangles and 1×1 squares. The rectangles can be oriented in either direction. How many ways can you do this?

2. Let **A** be the answer to problem number 1. Triangle ABC has a right angle at B and the length of AC is **A**. Let D be the midpoint of AB, and let P be a point in the plane such that $PA = PC = \frac{7\sqrt{5}}{4}$ and $PD = \frac{7}{4}$. The length of AB^2 is expressible as m/n for m, n relatively prime positive integers. Find m.

3. Let **B** be the answer to problem number 2. Let S be the set of positive integers less than or equal to **B**. What is the maximum size of a subset of S whose elements are pairwise relatively prime?

4. Let **C** be the answer to problem number 3. You have 9 shirts and 9 pairs of pants. Each is either red or blue, you have more red shirts than blue shirts, and you have same number of red shirts as blue pants. Given that you have **C** ways of wearing a shirt and pants whose colors match, find out how many red shirts you own.

5. Let **D** be the answer to problem number 4. You have two odd positive integers a, b. It turns out that $\text{lcm}(a, b) + a = \gcd(a, b) + b = \mathbf{D}$. Find ab.

6. Let **E** be the answer to problem number 5. A function f defined on integers satisfies $f(y) + f(12 - y) = 10$ and $f(y) + f(8 - y) = 4$ for all integers y. Given that $f(\mathbf{E}) = 0$, compute $f(4)$.

Solution: In this part problem 6 can be answered without knowing the answer to problem 5, and after that each problem can be done sequentially. For problem 6, put in $y = 4$ into the second equation to get $2f(4) = 4$, so $f(4) = 2$.

For problem 1, knowing $\mathbf{F} = 2$, condition on the number of rectangles. If there are none, then we have four squares for one way. If there is one, we can put it in any of four positions and tile the remaining area with squares for four ways. If there are two, they can be either horizontal or vertical for two ways. Summing up, we get an answer of 7.

For problem 2, knowing $\mathbf{A} = 7$, let E be the midpoint of AC. Since $PA = PC$, P is on the perpendicular bisector of AC, so $PE \perp AC$. We have $AE = \frac{7}{2}$, so using the Pythagorean theorem on $\triangle AEP$, we get $PE = \frac{7}{4}$. Then P is also on the perpendicular bisector of DE. Draw this perpendicular bisector and let it intersect AC at F; since $DE \perp AB$ we also have $AB \parallel PF$, and thus F is the midpoint of AE. Thus $PE = EF = \frac{7}{4}$, so $\triangle PEF$ is an isosceles right triangle and $\angle PFE = 45°$. Then also $\angle BAC = 45°$, so $\triangle ABC$ is also an isosceles right triangle. Thus $AB^2 = \frac{49}{2}$, so 49 is the answer.

For problem 3, knowing $\mathbf{B} = 49$, note that for any prime less than 49, that prime can divide at most one element of the set. So we can have 1 in the set and one element for each prime, but no more. There are 15 primes less than 49, so we can get up to 16 elements. This is achievable by taking 1 and every prime less than 49, so 16 is the answer.

For problem 4, knowing $\mathbf{C} = 16$, suppose we have r red shirts. We're given $r \geq 5$. Wearing matching red clothes has $r(9 - r)$ ways and matching blue clothes has $(9 - r)r$ ways. Since we need a total of 16 ways, $r(9 - r) = 8$, which has roots 1,8. Since $r \geq 5$, $r = 8$, the answer.

For problem 5, knowing $\mathbf{D} = 8$, let $a = dx$ and $b = dy$ for $d = \gcd(a, b)$ and x, y relatively prime. Note that $\text{lcm}(a, b) = dxy$. Then the equations become $dxy + dx = d + dy = 8$. So $x = 1$ and $8 = d(1 + y)$. We have a, b odd, so d is odd, meaning it must be 1. Thus $y = 7$, so $a = 1$, $b = 7$, and the answer is 7. This completes part 1.

Part II

7. Let $\mathbf{L}$ be the answer to problem number 12. You want to tile a nondegenerate square with side length $\mathbf{L}$ with 1×2 rectangles and 7×7 squares. The rectangles can be oriented in either direction. How many ways can you do this?

8. Let $\mathbf{G}$ be the answer to problem number 7. Triangle ABC has a right angle at B and the length of AC is $\mathbf{G}$. Let D be the midpoint of AB, and let P be a point in the plane such that $PA = PC = \frac{1}{2}$ and $PD = \frac{1}{2010}$. The length of AB^2 is expressible as m/n for m, n relatively prime positive integers. Find m.

9. Let $\mathbf{H}$ be the answer to problem number 8. Let S be the set of positive integers less than or equal to $\mathbf{H}$. What is the maximum size of a subset of S whose elements are pairwise relatively prime?

10. Let $\mathbf{I}$ be the answer to problem number 9. You have 391 shirts and 391 pairs of pants. Each is either red or blue, you have more red shirts than blue shirts, and you have same number of red shirts as red pants. Given that you have $\mathbf{I}$ ways of wearing a shirt and pants whose colors match, find out how many red shirts you own.

11. Let $\mathbf{J}$ be the answer to problem number 10. You have two odd positive integers a, b. It turns out that $\operatorname{lcm}(a,b) + 2a = 2\gcd(a,b) + b = \mathbf{J}$. Find ab.

12. Let $\mathbf{K}$ be the answer to problem number 11. A function f defined on integers satisfies $f(y) + f(7-y) = 8$ and $f(y) + f(5-y) = 4$ for all integers y. Given that $f(\mathbf{K}) = 453$, compute $f(2)$.

Solution: In this part problem 8 can be answered only by knowing the answer to problem 7 is a positive integer. By the triangle inequality, $AC \le PA + PC = 1$. Since AC is a positive integer, $AC = 1$ and P must be the midpoint of AC. Then $BC = 2 \cdot PD = \frac{1}{1005}$. Finally, $AB^2 = 1^2 - \frac{1}{1005^2} = \frac{1005^2 - 1}{1005^2}$, which has numerator 1010024, the answer.

If we attempt to do problem 9 using that $\mathbf{H} = 1010024$, we find that we need to know how many primes are less than 1010024. This is not feasible to do in a short answer timeframe. Instead, we must work backwards through the round, as we found that $\mathbf{G} = 1$ in the course of solving problem 8.

Given that the answer to problem 7 is 1, if the square has even side length then we can reflect any tiling of 1×2 rectangles about a diagonal to get a different tiling, so there must be more than one tiling if there are any. If it has an odd sidelength, it can be verified that there are no tilings for a side less than 7, there is one tiling for side length 7, and there is more than one tiling for side length greater than 7. So the side length must be 7 and $\mathbf{L} = 7$.

Given that the answer to problem 12 is 7, we have $f(2) = 7$. From the first equation $f(5) = 1$. Notice that by combining the two equations we get $4 = f(x+2) - f(x)$. Using this and what we know, $f(2k) = 3 + 4k$ and $f(2k+1) = -7 + 4k$ for all integers k. Since 453 is 1 mod 4, it is of the form $-7 + 4k$. Choosing $k = 115$, we get $f(231) = 453$, so $\mathbf{K} = 231$.

Given that the answer to problem 11 is 231, let $\gcd(a,b) = d$, $a = dx$, $b = dy$, and $\operatorname{lcm}(a,b) = dxy$ as in part 1. We have $xy + 2x = 2 + y$, or $(y+2)(x-1) = 0$. Since x, y are positive, $x = 1$. Now suppose $d > 1$. Then d^2 divides ab. But we know $ab = 231$ which has no square factors greater than 1, so d must be 1. Hence $a = 1$ and $b = 231$. This gives $\mathbf{J} = 233$.

Given that the answer to problem 10 is 233, we get that we have 233^2 ways to wear matching red clothes and 158^2 ways to wear matching blue clothes. Their sum is 79253, so $\mathbf{I} = 79253$.

This completes the round. One can indeed verify that the number of primes less than 1010024 is 79252, so problem 9 has also been done correctly.