Caltech Harvey Mudd Mathematics Competition

Tiebreaker Round **November 13, 2010**

1. The numbers 25 and 76 have the property that when squared in base 10, their squares also end in the same two digits. A positive integer is called amazing if it has at most 3 digits when expressed in base 21 and also has the property that its square expressed in base 21 ends in the same 3 digits. (For this problem, the last three digits of a one-digit number $\underline{b}$ are $00\underline{b}$, and the last three digits of a two-digit number $\underline{ab}$ are $0\underline{ab}$.) Compute the sum of all amazing numbers. Express your answer in base 21.

2. Let A, B, C, and D be points on a circle, in that order, such that $\overline{AD}$ is a diameter of the circle. Let E be the intersection of $\overleftrightarrow{AB}$ and $\overleftrightarrow{DC}$, let F be the intersection of $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$, and let G be the intersection of $\overleftrightarrow{EF}$ and $\overleftrightarrow{AD}$. If $AD = 8$, $AE = 9$, and $DE = 7$, compute EG.

3. Talithia throws a party on the fifth Saturday of every month that has five Saturdays. That is, if a month has five Saturdays, Talithia has a party on the fifth Saturday of that month, and if a month has four Saturdays, then Talithia does not have a party that month. Given that January 1, 2010 was a Friday, compute the number of parties Talithia will have in 2010.

4. Suppose a is a real number such that $3a + 6$ is the greatest integer less than or equal to a and $4a + 9$ is the least integer greater than or equal to a. Compute a.