Middle School JHMT 2020

Team Round

February 22, 2020

Instructions

- **<u>DO NOT</u> TURN OVER THIS PAPER UNTIL TOLD TO DO SO.**
- This test contains 20 questions to be solved in teams of up to 4 in 40 minutes.
- All answers, except that of the first question, will be integers.
- Only answers written on the appropriate area on the answer sheet will be considered for grading.
- Problems are weighted relative to their difficulty, determined by the number of students who solve each problem.
- No translators, books, notes, slide rules, calculators, abaci, or other computational aids are permitted. Similarly, graph paper, rulers, protractors, compasses, and other drawing aids are not permitted.
- If you believe the test contains an error, immediately tell your proctor and if necessary, report the error to the front desk after your exam.
- Good luck!

Use the following description of a machine to solve the first 4 problems in the round.

A machine displays four digits: 0000. There are two buttons: button A moves all digits one position to the left and fills the rightmost position with 0 (for example, it changes 1234 to 2340), and button B adds 11 to the current number, displaying only the last four digits if the sum is greater than 9999 (for example, it changes 1234 to 1245, and changes 9998 to 0009). We can denote a sequence of moves by writing down the buttons pushed from left to right. A sequence of moves that outputs 2100, for example, is $BABAA$.

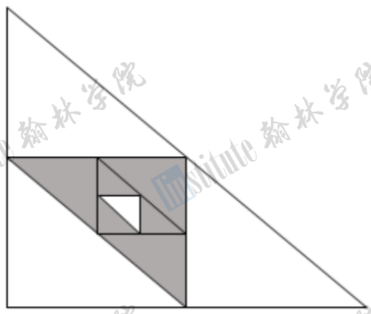
1. Give a sequence of 17 or less moves so that the machine displays 2020.

2. Using the same machine, how many outputs are possible if you make at most three moves?

3. Button B now adds n to the four digit display, while button A remains the same. For how many positive integers $n \leq 20$ (including 11) can every possible four-digit output be reached?

4. Suppose the function of button A changes to: move all digits one position to the right and fill the leftmost position with 2. Then, what is the minimum number of moves required for the machine to display 2020, if it initially displays 0000?

5. In the figure below, every inscribed triangle has vertices that are on the midpoints of its circumscribed triangle's sides. If the area of the largest triangle is 64, what is the area of the shaded region?



6. A bee flies $10\sqrt{2}$ meters in the direction 45° clockwise of North (that is, in the NE direction). Then, the bee turns 135° clockwise, and flies 20 forward meters. It continues by turning 60° counterclockwise, and flies forward 14 meters. Finally, the bee turns 120° clockwise and flies another 14 meters forward before finally finding a flower to pollinate. How far is the bee from its starting location in meters?

7. All the digits of a 15-digit number are either p or c. p shows up 3 more times than c does, and the average of the digits is $c - p$. What is $p + c$?

8. Let m be the sum of the factors of 75 (including 1 and 75 itself). What is the ones digit of m^{75}?

9. John flips a coin twice. For each flip, if it lands tails, he does nothing. If it lands heads, he rolls a fair 4-sided die with sides labeled 1 through 4. Let $\frac{a}{b}$ be the probability of never rolling a 3, in simplest terms. What is $a + b$?

10. Let $\triangle ABC$ have coordinates $(0, 0)$, $(0, 3)$ $(18, 0)$. Find the number of integer coordinates **interior** (excluding the vertices and edges) of the triangle.

11. What is the greatest integer k such that 2^k divides the value $20! \times 20^{20}$?

12. David has n pennies, where n is a natural number. One apple costs 3 pennies, one banana costs 5 pennies, and one cranberry costs 7 pennies. If David spends all his money on apples, he will have 2 pennies left; if David spends all his money on bananas, he will have 4 pennies left; is David spends all his money on cranberries, he will have 6 pennies left. What is the **second** least possible amount of pennies that David can have?

13. Elvin is currently at Hopperville which is 40 miles from Waltimore and 50 miles from Boshington DC. He takes a taxi back to Waltimore, but unfortunately the taxi gets lost. Elvin now finds himself at Kinsville, but he notices that he is still 40 miles from Waltimore and 50 miles from Boshington DC. If Waltimore and Boshington DC are 30 miles apart, What is the maximum possible distance between Hopperville and Kinsville?

14. After dinner, Rick asks his father for 1000 scoops of ice cream as dessert. Rick's father responds, "I will give you 2 scoops of ice cream, plus 1 additional scoop for every ordered pair (a, b) of real numbers satisfying $\frac{1}{a+b} = \frac{1}{a} + \frac{1}{b}$ you can find." If Rick finds every solution to the equation, how many scoops of ice cream will he receive?

15. Esther decides to hold a rock-paper-scissors tournament for the 56 students at her school. As a rule, competitors must lose twice before they are eliminated. Each round, all remaining competitors are matched together in best-of-1 rock-paper-scissors duels. If there is an odd number of competitors in a round, one random competitor will not compete that round. What is the maximum number of matches needed to determine the rock-paper-scissors champion?

16. $ABCD$ is a rectangle. X is a point on $\overline{AD}$, Y is a point on $\overline{AB}$, and N is a point outside $ABCD$ such that $XYNC$ is also a rectangle and $\overline{YN}$ intersects $\overline{BC}$ at its midpoint M. $m\angle BYM = 45°$. If $MN = 5$, what is the sum of the areas of $ABCD$ and $XYNC$?

17. Mr. Brown has 10 identical chocolate donuts and 15 identical glazed donuts. He knows that Amar wants 6 donuts, Benny wants 9 donuts, and Callie wants 9 donuts. How many ways can he distribute out his 25 donuts?

18. When Eric gets on the bus home, he notices his 12-hour watch reads 03:30, but it isn't working as expected. The second hand makes a full rotation in 4 seconds, then makes another in 8 seconds, then another in 12 seconds, and so on until it makes a full rotation in 60 seconds. Then it repeats this process, and again makes a full rotation in 4 second, then 8 seconds, etc. Meanwhile, the minute hand and hour hand continue to function as if every full rotation of the second hand represents 60 seconds. When Eric gets off the bus 75 minutes later, his watch reads $AB{:}CD$. What is $A + B + C + D$?

19. Alex and Betty want to meet each other at the airport. Alex will arrive at the airport between 12:00 and 13:15, and will wait for Betty for 15 minutes before he leaves. Betty will arrive at the airport between 12:30 and 13:10, and will wait for Alex for 10 minutes before she leaves. The chance that they arrive at any time in their respective time intervals is equally likely. The probability that they will meet at the airport can be expressed as $\frac{a}{b}$ where $\frac{a}{b}$ is a fraction written in simplest form. What is $a + b$?

20. Let there be $\triangle ABC$ such that $A = (0, 0), B = (23, 0), C = (a, b)$. Furthermore, D, the center of the circle that circumscribes $\triangle ABC$, lies on $\overline{\text{AB}}$. Let $\angle CDB = 150°$. If the area of $\triangle ABC$ is $\frac{m}{n}$ where m, n are in simplest integer form, find the value of $m \mod n$ (The remainder of m divided by n).

Team Round Solutions

1. The only solution that is 16 moves or less is $\boxed{BBBABBBBBBBBABBA}$. A breakdown of the solution is shown in the table below.

Move(s)	Display Afterwards
BBB	0033
A	0330
$BBBBBBBB$	0418
A	4180
BB	4202
A	2020

To see that this is the only solution, we can break the solutions down to two cases. If the last step is B, then the machine has to reach 2009 before that. The only way possible to reach a number ending with 9 is through 9 steps of B, so the number before that is 1910. This number is impossible to reach in less than 7 steps. Therefore the last step must be A, so the machine must reach $X202$ before that for some integer X.

By a similar process of reasoning, we see that the moves before that must be 2 or 12 steps of B. If we consider 12 steps of B, then the number before the machine reaches before that is $X070$, which is impossible to reach with less than 4 moves. Therefore we must have 2 steps of B, so the machine must reach $X180$ before that. Adding another move of A, the machine has to reach $YX18$ before that, in less than 13 moves.

To reach a number ending with 8, one must end with 8 steps of B, which means we need to reach $YX18 - 88 = (YX - 1)30$. The step before that must be A, so we end up with the number $Z(YX - 1)3$, which we must reach in less than 4 moves. Clearly, the only way to do this is with 3 moves of B, and we get our solution after putting all the moves in reverse order.

2. One or more A's before any other sequence results in no change (ex. $ABA = BA$), which means all 1 length and 2 length sequences are incorporated in some 3 length sequence. Thus we only need to check if the sequences of length 3 have any duplicates. All sequences of length 3 are shown below. Since there are no duplicates, the answer is $2^3 = \boxed{8}$.

- $AAA \Rightarrow 0000$
- $AAB \Rightarrow 0011$
- $ABA \Rightarrow 0110$
- $ABB \Rightarrow 0022$
- $BAA \Rightarrow 1100$
- $BAB \Rightarrow 0121$
- $BBA \Rightarrow 0220$
- $BBB \Rightarrow 0033$

3. $\boxed{8}$. In order for every possible four digits to be reached, every number in one's digit has to be reached. The one's digit of any combination of A, B can be achieved by combination of B exclusively through removing the sequence in front until the last A (ex: $BBABBAB \rightarrow B$).

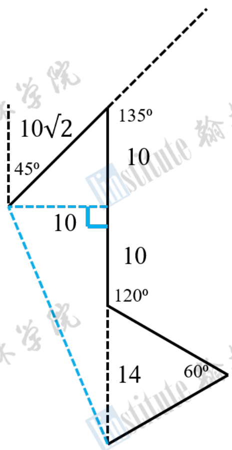
Let the GCD of n and 10 be k. If k is not 1, consecutive commands of B loops without reaching $k+1$ in the one's digit. Also, if k is 1, consecutive command of B of length p can result in 1 in the one's digit, making command of length np result in n in the one's digit. Thus n has to be co-prime with 10 to satisfy the given problem.

Furthermore, with similar reasoning above, if n is co-prime to $10,000$, then by using B exclusively, we can reach every possible four digit number. Luckily, numbers that are co-prime to $10,000$ is identical to numbers co-prime to 10. Thus the possible n are numbers that are co-prime to 10. There are 8 total such numbers: 1, 3, 7, 9, 11, 13, 17, 19.

4. $\boxed{11}$. The last step can either be A or B. If the last step is A, then the machine has to reach $020x$ before 2020, where x is an integer between 0 and 9. Since $020x$ starts with 0, it cannot be reached by any sequence of moves including A, since A would produce a number greater than 2000, and any sequence of Bs would only increase that number. Therefore the fastest way to reach a number $020x$ is by applying B 19 times, making 0209, so the total number of moves in this case is 20. If the last move is B, then the machine has to reach 2009. In order to reach 2009, the machine has to reach a number $009x$, by the same reasoning as before. This would take 9 moves of B, making 0099, so the total number of moves in this case is 11. The answer is therefore 11.

5. First, see that the largest triangle's area is 64. The edge length of the triangle inscribed inside the largest triangle is half that of the largest triangle. Thus, the triangle inscribed inside the largest triangle is $\frac{1}{4}$ the area of the largest triangle because both its base and the height will be $\frac{1}{2}$ the base and height of the large triangle. This relationship holds true for the next inscribed triangle, and so on. Thus, the area of the smallest white triangle in the middle is $64 \cdot \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{4} = 1$ and the black region has area of $64 \cdot \frac{1}{4} - 1 = \boxed{15}$.

6. After all the movements, the bee ends up 10 units East of his starting location, and 24 units south of his starting location. Thus, we can use the Pythagorean Theorem to conclude that he is $\sqrt{10^2 + 24^2} = \sqrt{676} = \boxed{26}$ units away from his starting location. Refer to the diagram below for clarity. The solid black line shows his path, the dotted black lines are to clarity his turning, and the blue line shows the calculation for the distance between his start and end locations.



7. From the given information, we are first trying to find values of p and c such that p and c appear a total of 15 times, and p occurs 3 more times than c. If we solve this linear equation, we find that p appears 9 times and c appears 6 times. Thus, we can set up the equation $\frac{9p + 6c}{15} = c - p$. If we multiple both sides by 15 and simply, we are left with the equation $8p = 3c$. Since p and c are both one-digit integers, the only solution to this equation is for $p = 3$, and $c = 8$. Thus, $p + c = \boxed{11}$.

8. We need to find the factors of 75. These factors are 1, 3, 5, 15, 25, and 75. Thus, the sum of the factors is 124. Note that since the prime factorization of 75 is 3×5^2, we can also find the sum of the factors of 75 using the expression $(3^0 + 3^1) \times (5^0 + 5^1 + 5^2)$. Note that this formula works because the if you distribute out the product, you end up with a direct sum of all the factors of 75. Also note that this formula works for finding the sum of the factors of any number!

   Now, we need to find the one's digit of 124^{75}. As any digit more significant than one's digit of 124 doesn't affect the one's digit of exponents of 124, this problem is same as finding the one's digit of 4^{75}. Notice that the one's digits of exponents of 4 alternate between 4 and 6. Thus, every odd exponent of 4 will have 4 as its one's digit, and every even exponent of 4 will have 6 as its one's digit. Thus, as 75 is odd, the answer is $\boxed{4}$.

9. In two coin flips, Jean has a $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$ chance of never having to roll the dice, a $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$ chance of having to roll the dice twice, and a $2 \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{2}$ chance of having to roll the dice once. Note that in this last case, we multiply by 2 because we can either roll a heads first then a tails, or a tails first then a heads.

   Furthermore, in each throw, there is $\frac{3}{4}$ chance that the die will result in 3. Thus the probability of not rolling a 3 is $\frac{1}{4} \cdot (\frac{3}{4})^0 + \frac{1}{2} \cdot (\frac{3}{4})^1 + \frac{1}{4} \cdot (\frac{3}{4})^2 = \frac{49}{64}$. Thus the answer is $\boxed{113}$.

10. Inside the rectangle $(0,0),(0,3),(18,0),(18,3)$, the number of interior integer coordinates is $2 \times 17 = 34$. Now draw a diagonal line segment with endpoints $(0,3),(18,0)$. The two triangles have identical number of interior integer coordinates. Also, two integer coordinates $(1,6),(2,12)$ are not in any of the two triangles. Thus the number of interior integer coordinates is $\frac{34-2}{2} = \boxed{16}$.

11. We are trying to find how many 2's are contained in the product $20! \times 20^{20}$. Let us start with the 20!. There are 10 numbers in this product that are divisible by 2, so we have ten 2's so far. Then, there are 5 numbers in this product that are divisible by 4, or 2^2. However, since these 5 numbers were already counted once earlier, we only count each of these 5 numbers once, so we have five more 2's here. Next, there are 2 numbers in this product that are divisible by 8. Again, note that since these 2 numbers were already counted twice earlier, we only count each of these 2 numbers once. Finally, there is 1 number in this product that is divisible by 16. Thus, we have $10+5+2+1 = 18$ as the total number of 2's in this product. In other words, this means that 2^{18} is the largest power of 2 that is a factor of 20!.

    Next, we repeat the same process with the second part of the product, 20^{20}. We see that a single 20 has 2^2 as a factor, which means that each 20 has two 2's. Thus, twenty 20's will have $20 \times 2 = 40$ 2's. As a whole, we have $18 + 40 = \boxed{58}$, as the answer.

12. First, if you have never solved a problem with the Chinese Remainder Theorem (CRT) before, Brilliant has a great page with background on the Theorem, linked here:
`https://brilliant.org/wiki/chinese-remainder-theorem/`. The solution below will use CRT to solve the problem. You may also want to read up on modular arithemtic if you have never used mods before to solve problems. In particular, here is a link with a good summary of performing modular division using multiplicative inverses:
`http://cs.brown.edu/courses/cs007/modmult/node2.html`.

    Since David will have 6 pennies left if he spends all his money on 7¢ cranberries, we can set up the equation $n = 7a+6$, where a is the maximum number of cranberries can be buy. If we do the same for bananas, we can set up the equation $n = 5b+4$, where b is the maximum number of bananas he can buy. Now, we have:

$$7a + 6 = 5b + 4$$
$$7a = 5b - 2$$
$$7a \equiv -2 \mod 5$$
$$7a \equiv 3 \mod 5$$
$$a \equiv 9 \mod 5$$
$$a \equiv 4 \mod 5$$

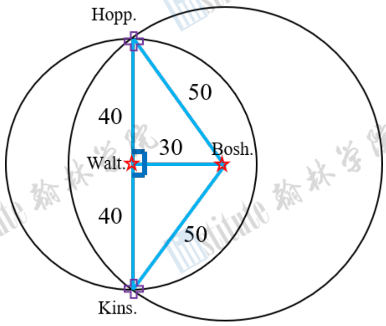
    Now, we can rewrite the last equation as $a = 5x + 4$ for some integer x. Now, we substitute this value of a back into the original equation of $n = 7a + 6$ to get $n = 35x + 34$. As this point, we have effectively set up an equation that represents possible values of n if we only consider the cranberry and banana constraints, i.e. 34, 69, 104 are all solutions that satisfy only the cranberry and banana constraints. To finish the problem, let's also consider the apples, whose constraint can be represented with the equation $n = 3c+2$, where c is the maximum number of apples David can buy. Thus, we have:

$$35x + 34 = 3c + 2$$
$$35x = -32 + 3c$$
$$35x \equiv -32 \mod 3$$
$$35x \equiv 1 \mod 3$$
$$x \equiv 2 \mod 3$$

    Now, we can rewrite the last equation as $x = 3y + 2$ for some integer y. Finally, we substitute this value of x back into the previous equation of $n = 35x + 34$ to get $n = 105y + 104$, which is a single equation that represents all possible values of n considering all three fruit constraints. Since n clearly must be positive (David cannot have a negative number of pennies), the smallest possible value of n is 104, and the second smallest is $105 + 104 = \boxed{209}$.

13. Refer to the figure below for clarity. Waltimore and Boshington DC are represented by the two red stars, and are placed 30 units apart. Then, a circle of radius 40 is drawn around Waltimore, and a circle of radius 50 is drawn around Boshington DC. The two points where the circles intersect indicate the two places that are exactly 40 units from Waltimore and 50 units from Boshington DC. These two points are labeled as Kinsville and Hopperville with the purple crosses.

    Once we draw this figure, we see that Hopperville, Boshington DC, and Waltimore can be thought of as vertices of a triangle, shown in blue in the figure. Notice these points form a right triangle since the side lengths are in the ratio $3-4-5$. Therefore, Kinsville, Waltimore, and Boshington DC also form a right triangle of the same side lengths. Hence, the distance between Elvin's two locations is simply $40+40=\boxed{80}$.

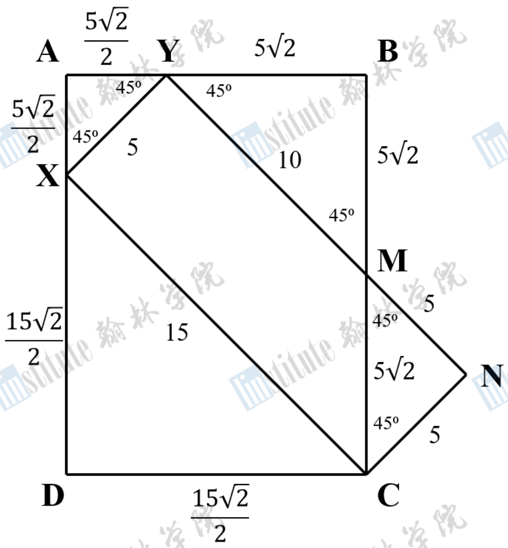


14. This equation simplifies to $(a+b)^2 = ab = -(a^2+b^2)$, or $a^2+b^2+(a+b)^2 = 0$, hence $a=b=0$. However, 0 cannot appear in the denominator, hence there are no real number ordered pairs (a,b) that solve the equation. Thus, Rick will only receive the initial $\boxed{2}$ scoops of ice cream.

15. While we could draw out a bracket and gradually count the number of matches that occur before a winner is found, this would be fairly tedious with 56 competitors. Instead, notice that whenever a match happens, one competitors has to lose. Furthermore, every competitor except the champion must lose twice before the champion can be crowned. Thus, for the 55 non-champion competitors, we need $55 \times 2 = 110$ matches in order to eliminate them all. Finally, since the problem asks for the maximum number of matches needed, notice it is also possible for the champion to lose once but still win. Thus, the maximum number of matches is $110+1=\boxed{111}$.

16. Refer to the diagram below for clarity. Since we know $m\angle BYM = 45°$, we can basically fill in every angle in the diagram as either a right angle or a 45° angle. Thus, since $MN = 5$, we also know that $NC = 5$, and since $\triangle MNC$ is a $45-45-90$ triangle, we also know that $MC = 5\sqrt{2}$. Furthermore, since M is the midpoint of $\overline{BC}$, we also find that $BM = 5\sqrt{2}$, which means the height of rectangle $ABCD$ is $10\sqrt{2}$. Continuing, since $\triangle YBM$ is also a $45-45-90$ triangle, $YB = 5\sqrt{2}$, and $XY = 5$ since it is the opposite side of NC. Since $\triangle AYX$ is also a $45-45-90$ triangle, $AY = AX = \frac{5\sqrt{2}}{2}$, which means the width of rectangle $ABCD$ is $\frac{15\sqrt{2}}{2}$, and the area of rectangle $ABCD$ is thus $\frac{15\sqrt{2}}{2} \times 10\sqrt{2} = 150$.

    We also know that the short side of rectangle $XYNC$ has length 5, and its long side has length $10+5=15$. Thus, the area of $XYNC$ is $5\times 15 = 75$. The sum of the areas of the two rectangles is thus $150+75=\boxed{225}$.



17. Since there are only two different types of donuts, we only have to focus on the number of ways we can distribute one of them, as whatever demand we don't satisfy with one donut must be satisfied by the other. Thus, let's only focus on the chocolate donuts. First, since Amar wants 6 donuts, there are 7 possibilities for number of chocolate donuts we can give Amar: anything from 0 to 6 inclusive. If Mr. Brown gives Amar 0 donuts, he then has 10 options for how many donuts to give Benny (anywhere from 0 to 9 inclusive), and he must give the rest to Callie. If Mr. Brown gives Amar 1 donut, he again has 10 options for how many donuts to give Benny. However, once Mr. Brown gives Amar 2 donuts, he only has 8 chocolate donuts left, and thus cannot give Benny 9 donuts, and only has 9 options for how many donuts to give Benny. We see that as Mr. Brown gives Amar more donuts, he has less options for how many donuts to give Benny. Thus, the total number of ways he can distribute his donuts can be expressed by the following sum: $10 + 10 + 9 + 8 + 7 + 6 + 5 = \boxed{55}$.

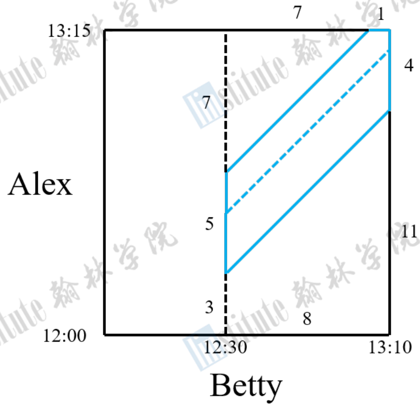
18. A full cycle of the second hand's rotation times takes a total of $4 + 8 + 12 + ... + 60 = 4 \times (1 + 2 + 3 + ... + 15) = 4 \times \frac{(1+15)(15)}{2} = 480$ seconds $= 8$ minutes. Since this full cycle results in the broken watch's minute hand advancing 15 minutes, we now know that 8 minutes passing in real times results in the broken watch advancing 15 minutes. Thus, in the 75 minutes of real time that pass while Eric is on the bus, $\left\lfloor \frac{75}{8} \right\rfloor = 9$ cycles of second hand occur, resulting in $9 \times 15 = 135$ minutes passing on the watch.

    Now, we have 3 minutes, or 180 seconds of real time left. We find that $4+8+12+...+36 = 180$, which represents 9 more minutes of real time. Thus, a total of 144 minutes pass on Eric's watch while he's on the bus, which means the time when he gets off is $05:54$, so the answer is $0 + 5 + 5 + 4 = \boxed{14}$.

19. Refer to the diagram below for clarity. This problem requires geometric probability. If you aren't comfortable with this topic, you can read about it here before moving on with this solution: `https://brilliant.org/wiki/1-dimensional-geometric-probability/`.

    In the diagram below, the y-axis represents the times that Alex can arrive, and the x-axis represents the times that Betty can arrive. Thus, for every point in the diagram, the x-coordinate of the point represents the time Betty would arrive, as represented by that point, and the y-coordinate of the point represents the time Alex would arrive, as represented by that point. Thus, in order for Alex and Betty to meet, Alex has to arrive at most 15 minutes before Betty, or Betty has to arrive at most 10 minutes before Alex. In the diagram below, the area surrounded by the blue lines represents the points where this condition is met. Thus, in order to find the probability of the two meeting, we divide the area surrounded by blue by the total area of the rectangle that represents all possible times the two can arrive. Note that the rectangle that represents all arrival times is only the right rectangle, as the left rectangle includes times that Betty cannot show up (times before 12:30).

In the diagram, the lengths given represent 5 minute intervals. In other words, the "length" of the segment connecting 12:30 and 13:10 is 8 units long because it represents $8 \times 5 = 40$ minutes. Note that while using "5 minutes" as a unit a measurement doesn't really make sense, it works here because all we care about is the ratio of the area enclosed by blue to the area of the rectangle representing all possible arrival times. Thus, units don't matter as long as we're consistent. Now, the area enclosed by blue is 39.5, and the area of the rectangle is 120, so the probability of a meeting is $\frac{39.5}{120} = \frac{79}{240}$, and the answer is thus $79 + 240 = \boxed{319}$.



20. As point D, the center of the circle that circumscribes D, lies on $\overline{\text{AB}}$, lines $\overline{\text{AD}}$, $\overline{\text{BD}}$, $\overline{\text{CD}}$ are radii of circle D. So $\overline{AD} = \overline{AD} = \overline{\text{CD}} = 23 \cdot \frac{1}{2} = \frac{23}{2}$. Let the foot of the triangle from vertex C be F. Then, as $\angle CDB = 150°$, $\angle CDA = 30°$. So $\triangle CFD$ is a $60°, 90°, 30°$ triangle. Thus $\overline{\text{CF}}$ is the half the length of CD. So $\overline{\text{CF}} = \frac{23}{2} \cdot \frac{1}{2} = \frac{23}{4}$ and $\triangle ABC = \frac{1}{2} \cdot \overline{AB} \cdot \overline{CF} = \frac{1}{2} \cdot 23 \cdot \frac{23}{4} = \frac{529}{8}$. As $529 = 8 \times 66 + 1$, the answer is $\boxed{1}$. Refer to the diagram below for more clarity!