Johns Hopkins Math Tournament 2020

Individual Round: Probability & Combinatorics

February 8, 2020

Instructions

- **DO NOT TURN OVER THIS PAPER UNTIL TOLD TO DO SO.**
- This test contains 12 questions to be solved individually in 60 minutes.
- All answers will be integers.
- Only answers written on the appropriate area on the answer sheet will be considered for grading.
- Problems are weighted relative to their difficulty, determined by the number of students who solve each problem.
- No translators, books, notes, slide rules, calculators, abaci, or other computational aids are permitted. Similarly, graph paper, rulers, protractors, compasses, and other drawing aids are not permitted.
- If you believe the test contains an error, immediately tell your proctor and if necessary, report the error to the front desk after the end of your exam.
- Good luck!

1. Elvin is trying to implement a digital passcode system for a classroom. The system requires the passcode to be a 4-digit number with distinct digits. It is known that the majority of passcodes start with 0, so Elvin decides to make the first digit not 0. He also decides that second digit cannot be 4, because 4 is an unlucky number in Chinese tradition. Let A be the number of possible passcodes where the first digit is not 0. Let B be the number of possible passcodes where the first digit is not 0 and the second digit is not 4. Find $|A - B|$.

2. In one variant of the Italian card game *Sette e Mezzo*, the deck contains cards 1 through 8 in each of 4 suits. After shuffling the deck, the probability that the 26^{th} card is a spade, the 30^{th} card is a club, and there is exactly 1 heart between those cards is $\frac{p}{q}$ with p, q coprime. Find $p + q$.

3. Maryland automobile license plates consist of 3 letters followed by 3 numbers. However, some 3-letter combinations are outlawed, often because they spell rude words. For each outlawed combination, how many possible license plates are removed from circulation?

4. How many permutations of the integers 1 to 7 have no adjacent even digits?

5. In the game of chess, the bishop may move and capture diagonally any number of squares in either direction. How many ways are there to place 14 bishops on an 8×8 chessboard so that none may capture any other?

6. Write a 5-digit number using only digits 1, 2, and 3. Do this such that each digit appears at least once. How many such numbers are possible?

7. 20 indistinguishable marbles are placed into 5 labeled sacks. However, there is a restriction that each sack cannot contain more than 10 marbles. Find the number of ways the marbles can be placed into the 5 sacks.

8. Suppose 6 points A, B, C, D, E, F are chosen uniformly at random on the circumference of some circle. The probability that line segments $\overline{AB}$, $\overline{CD}$, $\overline{EF}$ do not intersect is $\frac{p}{q}$ with p, q coprime. Compute $p + q$.

9. Arianna and Brianna are best friends and are in the same physics class. Their physics teacher assigns a lab project for students to work on in groups of 2 or 3. The class has 8 students, and the teacher randomly partitions the class into project groups such that all 385 possible partitions are equally likely. The probability that Arianna and Brianna end up in the same group can be expressed as $\frac{p}{q}$ with p, q coprime. Compute $p + q$.

10. In an interview process, 3 interviewers give integer scores ranging from 1 to 5. If the score from the first interviewer is a, the second is b, and the third is c, how many score sets (a, b, c) have sum $a + b + c$ greater than 7?

11. There are 5 different colored balls in a bag. 20 people each randomly choose a ball, record its color, and replace it. What is the expected number of pairs of people with the same recorded color? (n people with the same recorded color are counted as $\binom{n}{2}$ pairs.)

12. In a game of standard 3×3 tic-tac-toe, player X and player O take turns respectively placing an X or an O in an empty cell, with player X taking the first turn. Once one player places three markers collinearly on the board, that player wins, and the game ends. If rotations and reflections are considered distinct, how many ending configurations indicate that player O wins?

Probability & Combinatorics Solutions

1. There are 10 digits from 0 to 9, inclusive. To find A, the first digit can be everything but 0, and the rest of the digits can be anything but the ones before it. Thus, $A = 9 \cdot 9 \cdot 8 \cdot 7$. To compute B, consider the case where the first digit is 4 and the case where the first digit isn't 4. If the first digit is 4, the rest of the digits only have to be distinct from the previous ones, so there are $1 \cdot 9 \cdot 8 \cdot 7$ possibilities. If the first digit isn't 4, the first digit can't be 0 or 4, and the second digit can't be 4 or the first digit. The rest just have to be distinct from the first two, so there are $8 \cdot 8 \cdot 8 \cdot 7$ possibilities. The answer is $A - B = 9 \cdot 9 \cdot 8 \cdot 7 - (1 \cdot 9 \cdot 8 \cdot 7 + 8 \cdot 8 \cdot 8 \cdot 7) = \boxed{448}$.

2. The probability of the 26th card being a spade and the 30th card being a club is $\frac{8}{32} \cdot \frac{8}{31}$. Of the 27th, 28th, and 29th card, there are $\binom{3}{1}$ choices for the heart, then a $\frac{8}{30}$ chance it is a heart. The remaining two can be anything but a heart, with a probability of $\frac{22}{29} \cdot \frac{21}{28}$. The answer is
$$\frac{8}{32} \cdot \frac{8}{31} \cdot \binom{3}{1} \cdot \frac{8}{30} \cdot \frac{22}{29} \cdot \frac{21}{28} = \frac{132}{4495} \to \boxed{4627}.$$

3. Only the numbers need to be determined, so there are $10^3 = \boxed{1000}$ choices.

4. We can count 10 possible positions for the even numbers, which are 135, 136, 137, 146, 147, 157, 246, 247, 257, and 357. For each set of locations for the even numbers, there are 3! permutations of the even numbers and 4! permutations of the odd numbers, giving us an answer of $10 \cdot 3! \cdot 4! = \boxed{1440}$.

5. Tile the chessboard black and white in a standard manner by ensuring no black tiles share a side with another. There are either 7 or 8 diagonals formed by each color depending on the direction (i.e. northwest vs. northeast). Thus, there cannot be more than 7 bishops per color, so there are exactly 7 bishops per color. When placing bishops along the diagonals starting from the diagonal with two squares, the position of one bishop decides the position of the corresponding "opposite" bishop. There ends up being 2 choices for each of the four pairs of corresponding bishops, so there are $2^4 = 16$ choices for each color. For the two colors, the answer is $16^2 = \boxed{256}$.

6. Such a number must have either two of two different numbers or three of one number in the set $1, 2, 3$. In the former case, we choose 2 digits to be repeated, and we have $\frac{5!}{2!2!}$ ways to permute the numbers. In the latter case, we choose 1 digit to be repeated, and we have $\frac{5!}{3!}$ ways to permute the numbers. Our answer is
$$\binom{3}{2} \cdot \frac{5!}{2! \cdot 2!} + \binom{3}{1} \cdot \frac{5!}{3!} = \boxed{150}$$

7. Find the answer, with no restrictions, then subtract off the number of configurations that violate the size capacity. This is equal to
$$\binom{20+5-1}{20} - 5\binom{9+5-1}{9} = \binom{24}{20} - 5\binom{13}{9} = \boxed{7051}.$$

8. Only the groupings of the six points around the circle into pairs matters - their actual location is unimportant. There are $\frac{1}{6}\binom{6}{2,2,2} = 15$ possible groupings of the points into three unlabeled pairs, and five groupings result in no intersections. Thus the probability is $\frac{5}{15} = \frac{1}{3}$, so $p + q = 1 + 3 = \boxed{4}$.

9. The groupings are either 4 groups of 2 (Case X), or 2 groups of 3 and 1 group of 2 (Case Y). There are $\frac{\binom{8}{2}\cdot\binom{6}{2}\cdot\binom{4}{2}}{4!} = 105$ number of cases for Case X, and there are $\frac{\binom{8}{2}\cdot\binom{6}{3}}{2} = 280$ number of cases for Case Y. For case X, the chance the girls will be in the same group is $\frac{1}{7}$. To see why, consider Arianna to be in any group WLOG. There are only 7 possible slots for Brianna, and only 1 that will allow her to be in the same group as Arianna. For case Y, the chance the girls will be in the same group is $\frac{1}{4}$. Arianna has a $\frac{2}{8}$ chance of being in a group with only 2 people. Brianna will thus only have a 1 out of 7 chance to remain in the same group. If Arianna is in a group with 3 people, with a $\frac{6}{8}$ chance, then Brianna

will only have a 2 out of 7 chance to remain in the same group. Therefore, if the probability that both girls will be in the same group if Case Y were true is $\frac{1}{4}$.

Thus, the answer is
$$\frac{105}{385} \cdot \frac{1}{7} + \frac{280}{385} \cdot \frac{1}{4} = \frac{17}{77} \to \boxed{94}.$$

10. Out of $5^3 = 125$ total possible scores, only a few combinations add up to 7 or less. Up to ordering, they are 511, 421, 331, 322, 411, 321, 222, 311, 221, 211, 111. Accounting for ordering, there are 35 possible score combinations. Then the final answer is $125 - 35 = \boxed{90}$.

11. Let I_{ij} be the indicator function for the event that person i and person j match colors. $P(I_{ij} = 1) = \frac{1}{5}$. There are $\binom{20}{2}$ such indicators. Then the expected number is $\frac{1}{5}\binom{20}{2} = \boxed{38}$.

12. Player O can win when the board has either three Xs and three Os or four Xs and four Os. Also, three Os must be collinear, but no three of the Xs can be collinear.

    We begin with all cases with three Os. For each configuration of the Os, we consider the $\binom{6}{3} = 20$ ways player X could have played and disallow any winning positions for X. If the three Os form a horizontal or vertical line, then there are two rows that X could have won with, so we have $20 - 2 = 18$ ways to position the Xs in each case; we have six cases of horizontal/vertical wins for O, so we count $18 \cdot 6$ configurations. If the three Os form a diagonal line, then there are no rows for X to fill, so we have 20 ways to position the Xs in each case; there are two diagonals O could have won on, so we count $20 \cdot 2$ configurations.

    Next, we consider an ending with four Os and four Xs. For each configuration of the Os, we consider the $\binom{5}{4} = 5$ ways player X could have played and disallow any winning positions for X. Suppose that three Os lie in a horizontal or vertical row. Then, the fourth O is placed in a different, parallel row. Player X could have won by placing three Xs in the remaining parallel row and the fourth X in one of two spaces, so we have two combinations to disallow for each case, leaving $5 - 2 = 3$ acceptable states. There are six winning horizontal/vertical rows for O, and the fourth O marker can be placed in any one of six remaining slots, so we count $3 \cdot 6 \cdot 6$ configurations. Lastly, suppose that O wins on a diagonal. No matter where the fourth O lies, X has no winning positions. We have two possible diagonals, six remaining slots for the fourth O, and five ways to place the Xs, giving $5 \cdot 6 \cdot 2$ configurations.

    Altogether, we counted $18 \cdot 6 + 20 \cdot 2 + 3 \cdot 6 \cdot 6 + 5 \cdot 6 \cdot 2 = 108 + 40 + 108 + 60 = \boxed{316}$ ending configurations.