Johns Hopkins Math Tournament 2020

Individual Round: Calculus

February 8, 2020

Instructions

- **<u>DO NOT</u> TURN OVER THIS PAPER UNTIL TOLD TO DO SO.**
- This test contains 12 questions to be solved individually in 60 minutes.
- All answers will be integers.
- Only answers written on the appropriate area on the answer sheet will be considered for grading.
- Problems are weighted relative to their difficulty, determined by the number of students who solve each problem.
- No translators, books, notes, slide rules, calculators, abaci, or other computational aids are permitted. Similarly, graph paper, rulers, protractors, compasses, and other drawing aids are not permitted.
- If you believe the test contains an error, immediately tell your proctor and if necessary, report the error to the front desk after the end of your exam.
- Good luck!

Note: Euler's constant is $e = 2.71828\ldots$. Also, $\pi = 3.14159\ldots$. You will not need any more decimal places.

1. Let $f(x) = x^2 + e^{x\sqrt{2}}$. Compute the maximum possible value of $\frac{f''''(x)}{f(x)}$ over all real x.

2. Compute
$$\lim_{x\to\infty}\left(\sqrt{x^2+120x+121}-\sqrt{x^2+20x+21}\right).$$

3. For a certain choice of relatively prime positive integers a and b, the function $f(x) = e^{x\sqrt{3}}\sin x$ is increasing on the interval $(0, a\pi/b)$ and decreasing on the interval $(a\pi/b, \pi)$. Compute $a+b$.

4. Compute the greatest integer less than or equal to
$$\int_{\pi-1}^{\pi+1} \frac{e^{\cos x}\cos^3 x + \cot x}{\cot x}\,dx.$$

5. Four robots live on the Euclidean plane and are represented by the points R_0, R_1, R_2, and R_3. Initially, quadrilateral $R_0R_1R_2R_3$ is a square of side length 100. Then, for $i = 0, 1, 2,$ and 3, robot R_i chases robot $R_{(i+1) \bmod 4}$ such that, at any moment, all four robots are moving at equal speed and move directly toward their respective targets. In this manner, the robots' paths spiral toward the center of the original square. Compute the smallest positive integer L such that each of these paths cannot have a length greater than L.

6. For $p \geq 1$ and a vector $\vec{x} = \langle x_1, x_2, \ldots, x_n\rangle$ of n real numbers, define $\|\vec{x}\|_p := \left(\sum_{k=1}^n |x_k|^p\right)^{1/p}$, and define $\|\vec{x}\|_\infty := \lim\limits_{p\to\infty}\|\vec{x}\|_p$. Let $\vec{u} = \langle u_1, u_2, \ldots, u_{10}\rangle$, where $u_k = 2020 + 202k - 20k^2$ for $k = 1, 2, \ldots, 10$. What is the value of $\|\vec{u}\|_\infty$?

7. The first (i.e., leftmost) nonzero digit of the base-ten expansion of $999999^{1000000}$ is A, and the next two digits are B and C in order from left to right. Compute the value of $100A + 10B + C$.

8. Let $f(x) = e^{-1/x}$ be defined over positive real values of x. Find the smallest integer k such that
$$\lim_{x\to 0^+} \frac{x^k f^{(2020)}(x)}{f(x)} = 0,$$
where $f^{(2020)}(x)$ denotes the 2020th derivative of f evaluated at x.

9. Determine the largest integer n such that $2\sqrt[4]{5} + \frac{1}{n} > 3$.

10. Compute the value of $10a + b$, where a and b are positive integers satisfying
$$\int_0^\pi \frac{x\sin x}{1+\cos^2 x}\,dx = \frac{\pi^b}{a}.$$

11. A polynomial $P(x)$ of some degree d satisfies $P(0) = 1$ and $P(n) = n^2 + 20n - 20$ and $P'(n) = 2n + 40$ for $n = 1, 2, 3, 4, 5$. Also, P has d distinct (not necessarily real) roots $r_1, r_2, \ldots, r_d$. The value of
$$\sum_{k=1}^{d} \frac{1}{3 - r_k}$$
can be expressed as a common fraction $\frac{m}{n}$. What is the value of $m+n$?

12. What is the greatest integer less than or equal to
$$100\sum_{n=1}^{100} 3^n \sin^3\left(\frac{\pi}{3^n}\right)?$$

Calculus Solutions

1. The fourth derivative of f is $f''''(x) = \sqrt{2}^4 e^{x\sqrt{2}} = 4e^{x\sqrt{2}}$, so we wish to maximize $\frac{4e^{x\sqrt{2}}}{x^2+e^{x\sqrt{2}}}$. Using the fact $x^2 \geq 0$, with equality only at $x = 0$, we conclude

$$\frac{f''''(x)}{f(x)} = \frac{4e^{x\sqrt{2}}}{x^2 + e^{x\sqrt{2}}} \leq \frac{4e^{x\sqrt{2}}}{e^{x\sqrt{2}}} = 4,$$

with equality if and only if $x = 0$. The answer is therefore $\boxed{4}$.

2. Let $u = 1/x$ so that

$$\begin{aligned}\lim_{x\to\infty}\left(\sqrt{x^2+120x+121} - \sqrt{x^2+20x+21}\right) &= \lim_{x\to\infty} x\left(\sqrt{1+120/x+121/x^2} - \sqrt{1+20/x+21/x^2}\right) \\ &= \lim_{u\to 0}\frac{\sqrt{1+120u+121u^2} - \sqrt{1+20u+21u^2}}{u}.\end{aligned}$$

By L'Hopital's rule, we differentiate the numerator and denominator of the fraction and rewrite the limit as

$$\lim_{u\to 0}\frac{\frac{120+242u}{2\sqrt{1+120u+121u^2}} - \frac{20+42u}{2\sqrt{1+20u+21u^2}}}{1}.$$

The resulting expression is well defined at $u = 0$, so we directly plug in $u = 0$ to conclude that the limit equals $\boxed{50}$.

3. If f is increasing on $(0, a\pi/b)$ and decreasing on $(a\pi/b, \pi)$, then $a\pi/b$ is a local maximizer of f in $(0, \pi)$, meaning $f'(a\pi/b) = 0$ and $f''(a\pi/b) \leq 0$. We have $f'(x) = e^{x\sqrt{3}}(\cos x + \sqrt{3}\sin x)$, so we need $\cos\frac{a\pi}{b} + \sqrt{3}\sin\frac{a\pi}{b} = 0 \implies \tan\frac{a\pi}{b} = -\frac{1}{\sqrt{3}}$. We conclude $a\pi/b = 5\pi/6$, so $a + b = 5 + 6 = \boxed{11}$. To verify that $5\pi/6$ is a local *maximizer*, we note that $x \in (0, 5\pi/6) \implies f'(x) > 0$ and $x \in (5\pi/6, \pi) \implies f'(x) < 0$.

4. Rewrite the integral as $\int_{\pi-1}^{\pi+1}\left(e^{\cos x}\cos^2 x\sin x + 1\right)dx = 2 + \int_{\pi-1}^{\pi+1} e^{\cos x}\cos^2 x\sin x\, dx$. Since $\cos x$ is an even function about $x = \pi$ and $\sin x$ is an odd function about $x = \pi$, if we let $f(x) = e^{\cos x}\cos^2 x\sin x$, then $f(2\pi - x) = -f(x)$ and thus

$$2 + \int_{\pi-1}^{\pi+1} f(x)dx = 2 + \int_{\pi-1}^{\pi}\left(f(x) + f(2\pi - x)\right)dx = 2 + \int_{\pi-1}^{\pi} 0\, dx = \boxed{2}.$$

5. There is a methodical calculus-based solution to this problem that starts with the robots moving in discrete time steps and then considers the limit of their path length as the time step goes to zero. However, there is an elegant solution to this problem that avoids calculus! This solution focuses on the idea of a chaser "gaining ground" on their target.

   At any point in time, the four robots' locations form the vertices of a square, and adjacent sides of a square are perpendicular. Also, at any moment, the vector $\overrightarrow{R_iR_{(i+1) \bmod 4}}$ has the same orientation as the velocity of robot R_i, who is moving directly toward robot $R_{(i+1) \bmod 4}$. Therefore, if robot A is directly chasing robot B, then A's velocity is orthogonal (perpendicular) to B's velocity at every point in time. Robot B's velocity is then also perpendicular to vector $\overrightarrow{AB}$, so B has a zero velocity component in the direction of $\overrightarrow{AB}$. This means that B's movement is not responsible for B getting any closer to or farther from A, so all the ground that A gains on B is done so purely by A's movement. Thus, the length of A's pursuit of B is the same as what it would have been if B had remained stationary. For any pair of robots (A, B) with A chasing B, robot A is initially 100 units away from B, so as the paths of the four robots converge in the center of the square, A will get arbitrarily close to B, so A's path length will get arbitrarily close to $\boxed{100}$, without exceeding this length.

6. This problem deals with the so-called L^p-norm and infinity-norm. For an n-entry nonzero vector $\vec{x}$, if we let $m = \max_{1\le k\le n} |x_k|$, then

$$\|\vec{x}\|_\infty = \lim_{p\to\infty}\left(\sum_{k=1}^{n}|x_k|^p\right)^{1/p} = \lim_{p\to\infty}\left(m^p\sum_{k=1}^{n}\left|\frac{x_k}{m}\right|^p\right)^{1/p} = m\lim_{p\to\infty}\left(\sum_{k=1}^{n}\left|\frac{x_k}{m}\right|^p\right)^{1/p}.$$

Because there exists an index $k^\star$ that satisfies $\left|\frac{x_{k}^\star}{m}\right| = 1$, we have $\sum_{k=1}^n \left|\frac{x_k}{m}\right|^p \ge 1$. By definition of m, we also know $0 \le \left|\frac{x_k}{m}\right| \le 1$ for all indices k, so $\sum_{k=1}^n \left|\frac{x_k}{m}\right|^p \le \sum_{k=1}^n 1^p = n$. By the squeeze theorem,

$$1 \le \sum_{k=1}^{n}\left|\frac{x_k}{m}\right|^p \le n \implies \lim_{p\to\infty} 1^{1/p} \le \lim_{p\to\infty}\left(\sum_{k=1}^{n}\left|\frac{x_k}{m}\right|^p\right)^{1/p} \le \lim_{p\to\infty} n^{1/p}$$

$$\implies 1 \le \lim_{p\to\infty}\left(\sum_{k=1}^{n}\left|\frac{x_k}{m}\right|^p\right)^{1/p} \le 1 \implies \lim_{p\to\infty}\left(\sum_{k=1}^{n}\left|\frac{x_k}{m}\right|^p\right)^{1/p} = 1.$$

Therefore, $\|\vec{x}\|_\infty = m$, so to compute $\|\vec{u}\|_\infty$, we just need to find the maximum value of $|u_k|$ over all indices k from 1 to 10. The quadratic $2020 + 202k - 20k^2$ has a global maximum at $k = \frac{202}{2\cdot 20} = 5.05$, so $k = 5$ maximizes the value of u_k over integers k. To show $k = 5$ also maximizes $|u_k|$, it suffices to show that $u_k > 0$ for all k from 1 to 10. Of the integers $1, 2, \ldots,$ and 10, setting $k = 10$ makes k the farthest possible from the global maximum at 5.05. Then, the smallest value of u_k will be $u_{10} = 2020 + 202\cdot 10 - 20\cdot 10^2 = 4040 - 2000 = 2040 > 0$. All entries of $\vec{u}$ are positive, so $\|\vec{u}\|_\infty = u_5 = 2020 + 202\cdot 5 - 20\cdot 5^2 = 3030 - 500 = \boxed{2530}$.

7. Rewrite 1000000 as 10^6, so that $999999 = 10^6 - 1$. Then, $999999^{1000000} = \left(10^6-1\right)^{10^6} = \left(10^6\right)^{10^6}\left(1-10^{-6}\right)^{10^6}$. Multiplying a decimal number by $10^{(6\cdot 10^6)}$ just shifts the decimal point $6\cdot 10^6$ places to the right, without changing the sequence of digits comprising the number. Therefore, the leading digits of interest (A, B, and C) are the leading digits in the decimal expansion of $\left(1-10^{-6}\right)^{10^6}$, which can be rewritten as $(1-1/n)^n$ for $n = 10^6$. This value of n is pretty large, and we know that $\lim_{n\to\infty}(1+k/n)^n = e^k$. Therefore, $(1-1/n)^n$ will get super close to $1/e$ as n gets large, and $n = 10^6$ is large enough for $(1-1/n)^n$ to approximate $1/e$ correctly to three decimal places. In other words, A, B, and C are the leading three digits of the decimal expansion of $1/e$. By plugging in $e = 2.71828\ldots$ and doing some long division, we will calculate $1/e = 0.3678\ldots$, so $100A + 10B + C = \boxed{367}$.

8. By the Chain Rule, $f'(x) = e^{-1/x}/x^2 = f(x)x^{-2}$. By the Product Rule, $f''(x) = f'(x)x^{-2} - 2f(x)x^{-3} = f(x)x^{-4} - 2f(x)x^{-3}$. In general, the derivative of $f(x)x^{-n}$ is $f'(x)x^{-n} - nf(x)x^{-(n+1)} = f(x)x^{-(n+2)} - nf(x)x^{-(n+1)}$. Therefore, if we let $g_n(x) = f(x)x^{-n}$, then any repeated derivative $f^{(z)}(x)$ can be expressed as a linear combination of the functions $g_n(x)$ over $n \in \mathbb{N}$. By writing out the first, second, and third derivatives of f, we can readily observe that $2z$ is the largest value of n such that $g_n(x)$ is present in the linear combination expansion of $f^{(z)}(x)$. This observation can be verified with a more rigorous inductive proof. Then, $f^{(2020)}(x)$ can be written as a linear combination of terms in $\{g_1(x), g_2(x), \ldots, g_{4040}(x)\}$, with a nonzero coefficient of $g_{4040}(x)$ in the linear combination. If we let this coefficient be c, then for some polynomial $P(x)$, we have

$$f^{(2020)}(x) = \frac{f(x)(c + xP(x))}{x^{4040}} \implies \frac{x^k f^{(2020)}(x)}{f(x)} = x^{k-4040}(c + xP(x)).$$

Thus, the limit $\lim_{x\to 0^+} \frac{x^k f^{(2020)}(x)}{f(x)}$ is finite only for $k \ge 4040$ and zero only for $k > 4040$. So to make the limit zero, the smallest option is $k = \boxed{4041}$.

9. Observe that $2\sqrt[4]{5} = \sqrt[4]{80}$, and $\sqrt[4]{80}$ is close to $\sqrt[4]{81}$, which is 3. We can algebraically manipulate the given inequality:

$$\sqrt[4]{80} + \frac{1}{n} > \sqrt[4]{81} \implies \frac{1}{n} > \sqrt[4]{81} - \sqrt[4]{80} \implies n < \frac{1}{\sqrt[4]{81} - \sqrt{480}} = \frac{(\sqrt{81}+\sqrt{80})(\sqrt[4]{81}+\sqrt[4]{80})}{81-80}$$

$$\implies n < (9+\sqrt{80})(3+\sqrt[4]{80}) < (9+9)(3+3) = 108.$$

We suspect $(9+\sqrt{80})(3+\sqrt[4]{80})$ to be very close to 108, and a reasonable guess at this point is $n = 107$. Let $f(x) = x^{1/2}$ and $g(x) = x^{1/4}$. Note that these functions are concave down, so

$$f'(81) < \frac{f(81)-f(80)}{81-80} < f'(80) \implies \frac{1}{18} = \frac{1}{2\sqrt{81}} < 9 - \sqrt{80} < \frac{1}{2\sqrt{80}} < \frac{1}{17} \text{ and}$$

$$g'(81) < \frac{g(81)-g(80)}{81-80} < g'(80) \implies \frac{1}{108} = \frac{1}{4 \cdot 81^{3/4}} < g(81) - g(80) < \frac{1}{4 \cdot 80^{3/4}} = \frac{1}{4 \cdot 2^3 \cdot 5^{3/4}} < \frac{1}{4 \cdot 8 \cdot 2} = \frac{1}{64}.$$

Then,

$$(9+\sqrt{80})(3+\sqrt[4]{80}) = (18-(9-\sqrt{80}))(6-(3-\sqrt[4]{80})) > \left(18 - \frac{1}{17}\right)\left(6 - \frac{1}{64}\right) > 108 - \frac{6}{17} - \frac{18}{64} > 107,$$

so $n = \boxed{107}$ is indeed the maximum feasible value of n.

10. Let $u = \cos x$ and $du = -\sin x \, dx$ so that

$$\int_0^\pi \frac{x \sin x}{1+\cos^2 x}\,dx = \int_1^{-1} \frac{\cos^{-1}(u) \cdot (-du)}{1+u^2} = \int_0^1 \frac{\cos^{-1}(-u) + \cos^{-1}(u)}{1+u^2}\,du.$$

Using the fact that $\cos^{-1}(-u) + \cos^{-1}(u) = \pi$, we observe that

$$\frac{\pi^b}{a} = \int_0^1 \frac{\cos^{-1}(-u)+\cos^{-1}(u)}{1+u^2}\,du = \pi \int_0^1 \frac{du}{1+u^2} = \pi \left[\tan^{-1} u\right]_0^1 = \pi\left(\frac{\pi}{4} - 0\right) = \frac{\pi^2}{4},$$

so we have $a = 4$, $b = 2$, and $10a + b = \boxed{42}$, the answer to life, the universe, and everything.

11. We can write $P(x)$ in the form $a\prod_{j=1}^d (x - r_j)$ for some scalar a. For any x that is not a root of P,

$$\frac{1}{x-r_k} = \frac{a\prod_{j\in\{1,\ldots,d\}\setminus\{k\}}(x-r_j)}{a\prod_{j=1}^d (x-r_j)} \implies \sum_{k=1}^d \frac{1}{x-r_k} = \frac{\sum_{k=1}^d a\prod_{j\in\{1,\ldots,d\}\setminus\{k\}}(x-r_j)}{a\prod_{j=1}^d(x-r_j)} = \frac{P'(x)}{P(x)},$$

where we used the product rule to recognize the expression for $P'(x) = \frac{d}{dx} a \prod_{k=1}^d (x - r_k)$. Therefore, the desired sum equals $\frac{P'(3)}{P(3)} = \frac{2\cdot 3+40}{3^2+20\cdot 3-20} = \frac{46}{49}$, so the answer is $46 + 49 = \boxed{95}$.

12. The sine triple-angle identity states that $\sin(3\theta) = 3\sin\theta - 4\sin^3\theta$, so if we let $a_n = 3^n \sin^3\left(\frac{\pi}{3^n}\right)$, then

$$\sin\left(\frac{\pi}{3^{n-1}}\right) = 3\sin\left(\frac{\pi}{3^n}\right) - \frac{4a_n}{3^n} \implies a_n = \frac{3}{4}\left(3^n \sin\left(\frac{\pi}{3^n}\right) - 3^{n-1}\sin\left(\frac{\pi}{3^{n-1}}\right)\right).$$

Therefore,

$$\sum_{n=1}^{100} a_n = \frac{3}{4}\left(3^{100}\sin\left(\frac{\pi}{3^{100}}\right) - \sin\pi\right) = \frac{3\pi}{4} \cdot \frac{\sin\left(\pi/3^{100}\right)}{\pi/3^{100}} \approx \frac{3\pi}{4},$$

where we used the engineer's approximation $\sin x \sim x$ for small x. This approximation has negligible error when $x = \pi/3^{100}$, which is very very small, so the answer is $\lfloor 100 \cdot (3\pi/4) \rfloor = \lfloor 75\pi \rfloor = \boxed{235}$.