Johns Hopkins Math Tournament 2020

Individual Round: Algebra and Number Theory

February 8, 2020

Instructions

- **<u>DO NOT</u> TURN OVER THIS PAPER UNTIL TOLD TO DO SO.**
- This test contains 12 questions to be solved individually in 60 minutes.
- All answers will be integers.
- Only answers written on the appropriate area on the answer sheet will be considered for grading.
- Problems are weighted relative to their difficulty, determined by the number of students who solve each problem.
- No translators, books, notes, slide rules, calculators, abaci, or other computational aids are permitted. Similarly, graph paper, rulers, protractors, compasses, and other drawing aids are not permitted.
- If you believe the test contains an error, immediately tell your proctor and if necessary, report the error to the front desk after the end of your exam.
- Good luck!

1. The roots of a quadratic equation $ax^2 + bx + c$ are 3 and 5, and the leading coefficient is 2. What is $a + b + c$?

2. If x is a real number satisfying $\frac{1}{\sqrt{x}} + \frac{2}{1+\sqrt{x}} = 2$, find the value of x. Note that $\sqrt{x}$ denotes the positive square root of x.

3. The sum of the squares of the reciprocals of the roots of the equation $x^3 + 2x^2 + 8x + 7 = 0$ can be expressed as $\frac{p}{q}$, where p and q are relatively prime. Find $p + q$.

4. Our base-ten number system is endowed with a neat rule for divisibility by 9: when an integer N is written in base-ten, N is divisible by 9 if and only if the sum of N's digits is divisible by 9. Compute the sum of all positive integers b_{10} between 11_{10} and 99_{10} such that any integer N is divisible by 9 if and only if the sum of N's digits in base b is divisible by 9.

5. Compute the value of
$$\sum_{n=2}^{2018} \frac{2}{1 + \log_n(2020 - n)}.$$

6. Find the last three digits of 99^{99}.

7. Let $a_1 = 3$, $a_2 = 8$, and $a_n = \sum_{k=1}^{n-1} a_k$ for $n > 2$. The value of $\sum_{n=1}^{\infty} \frac{1}{a_n}$ can be written as a common fraction $\frac{p}{q}$. Compute $p + q$.

8. What is the least number of weights required to weigh any integral number of pounds up to 360 pounds if one is allowed to put weights in both pans of a balance?

9. The equation $x^3 + 6x - 2 = 0$ has exactly one real solution, $x = \sqrt[3]{a} + \sqrt[3]{b}$, where a and b are integers not divisible by the cube of any prime. If $a > b$, then compute $100a + b$.

10. For all $x > 1$, the equation $x^{60} + kx^{27} \geq kx^{30} + 1$ is satisfied. What is the largest possible value of k?

11. Given a real number a_1, recursively generate a sequence $\{a_1, a_2, a_3, \ldots\}$ satisfying
$$a_{n+1} = -\frac{1}{a_n - 1} - \frac{1}{a_n + 1}$$
for all $n \in \mathbb{N}$. Out of all real numbers, how many values of a_1 result in the equality $a_6 = a_1$?

12. Let $f(n) = n^2 - 1$ and $g(n) = (n+1)^3 - (n-1)^3$. Let $\times$ be a binary operation that acts on two ordered pairs, defined by the following rule: $(a, b) \times (c, d) = (ac, ad + bc)$. For integers $n \geq 3$, let
$$(a_n, b_n) = [[[(f(2), g(2)) \times (f(3), g(3))] \times (f(4), g(4))] \times \cdots] \times (f(n), g(n)).$$
Determine the smallest n such that $b_n > 2020a_n$.

Algebra and Number Theory Solutions

1. As the two roots are 3,5, and the leading coefficient is 2, the equation could be written as $2(x-3)(x-5) = 2(x^2-8x+15) = 2x^2-16x+30$. So $a+b+c = 2-16+30 = \boxed{16}$.

2. First we can combine the two fractions by finding a common denominator, which is $(1+\sqrt{x})\sqrt{x} = \sqrt{x}+x$, giving us $\dfrac{1+3\sqrt{x}}{\sqrt{x}+x} = 2$. We can then multiply the denominator to the right side of the equation and subtract the right side of the equation from the left giving us $-2x+\sqrt{x}+1=0$. This is quadratic in $\sqrt{x}$ and we can thus apply the quadratic equation to get that $\sqrt{x}=1$. (The note allows us to ignore the negative solution) Thus $x = \boxed{1}$.

3. (Quickest solution) We can first generate a polynomial that has roots that equals the reciprocal of roots of the original equation by switching coefficients, doing that we get $7x^3+8x^2+2x^2+1$. Let the sum of the roots be P_1, and by Vieta's formula we get $P_1 = \frac{-8}{7}$. Let the sum of the squares of the roots be P_2. Using Newton's sum($7P_2+8P_1+(2)(2)=0$), we get $7P_2+8\frac{-8}{7}+2\cdot 2 = 0$,$P_2 = \frac{36}{49}$ Better solution: let the roots be a,b,c, we wish to find $(\frac{1}{a})^2+(\frac{1}{b})^2+(\frac{1}{c})^2$. This is equivalent to $\frac{(ab+ac+ab)^2-2(a+b+c)(abc)}{abc^2}$, which is $\frac{8^2-2\cdot(-2)\cdot(-7)}{(-7)^2} = \frac{36}{49}$. Thus, the answer is $36+49 = \boxed{85}$.

4. The rule for divisibility by 9 works in base ten because $10 \equiv 1 \mod 9$, so $10^k \equiv 1 \mod 9$ for any nonnegative integer k. The value of a generic n-digit whole number $\overline{a_{n-1}a_{n-2}\dots a_1a_0}_{10}$ in base ten is $a_{n-1}10^{n-1}+a_{n-2}10^{n-2}+\dots+a_110^1+a_010^0$, which is equivalent to $a_{n-1}\cdot 1+a_{n-2}\cdot 1+\dots+a_1\cdot 1+a_0\cdot 1$ modulo 9, hence the neat divisibility rule. The same trick works in base b if and only if $b \neq 1$ and $b \equiv 1 \mod 9$. Thus, the sum of all such b between 11_{10} and 99_{10} is $19+28+37+\dots+91 = \sum_{k=2}^{10}(9k+1) = 9\sum_{k=1}^{10} k = \frac{9\cdot 10\cdot 11}{2} = \boxed{495}$.

5. The essential observation is $\log_a(b)\cdot\log_b(a) = 1$. Therefore, if we let $f(n) = \log_n(2020-n)$, then $f(2020-n) = \frac{1}{f(n)}$, so

$$\sum_{n=2}^{2018}\frac{2}{1+f(n)} = \sum_{n=2}^{2018}\left(\frac{1}{1+f(n)}+\frac{1}{1+f(2020-n)}\right) = \sum_{n=2}^{2018}\left(\frac{1}{1+f(n)}+\frac{1}{1+1/f(n)}\right) = \sum_{n=2}^{2018}\frac{1+f(n)}{1+f(n)}$$
$$= \sum_{n=2}^{2018}(1) = 2018-2+1 = \boxed{2017}.$$

6. We can write 99^{99} as $(100-1)^{99}$ and do binomial expansion, so the equation becomes $\binom{99}{0}\cdot 100^0\cdot(-1)^{99}+\binom{99}{1}\cdot 100^1\cdot(-1)^{98}+\dots+\binom{99}{99}\cdot 100^{99}\cdot(-1)^0$. But the only items that affect the last three digits are the first two items. A simple computation gives us the answer $\boxed{899}$.

7. Note that $a_3 = 11$. For integers $n \geq 4$, $a_n = a_{n-1}+\sum_{k=1}^{n-2} a_k = a_{n-1}+a_{n-1} = 2a_{n-1}$. Hence, $a_n = 2^{n-3}a_3$ for $n \geq 3$, so

$$\sum_{n=1}^{\infty}\frac{1}{a_n} = \frac{1}{3}+\frac{1}{8}+\sum_{n=3}^{\infty}\frac{1}{2^{n-3}\cdot 11} = \frac{11}{24}+\frac{1}{11}\sum_{k=0}^{\infty}\frac{1}{2^k} = \frac{11}{24}+\frac{2}{11} = \frac{169}{264}.$$

Hence, the answer is $169+264 = \boxed{433}$.

8. A quick way to gain some insight is to realize that the setting can be understood as base 3 representation. Using the 6 weights $3^5, 3^4, 3^3, 3^2, 3^1, 3^0$, the maximum number we can represent is 364, so they are enough to represent until 360. To prove this is the least number, suppose we have only 5 weights, then because each weight has 3 places to go to: the right side of the pan, the left side, or not on the balance, the maximum number of outcomes is $(3^5-1)/2 = 121$, smaller than 360. So the answer is $\boxed{6}$.

9. For completeness, we should first justify that, if p and q are irrational cube roots of rational numbers, then $p+q$ cannot be nonzero and rational. Suppose that $p+q=r$ for some $r \in \mathbb{Q} \setminus \{0\}$. Consider the identity
$$p^3+q^3-r^3+3pqr=(p+q-r)\left(p^2+q^2+r^2-pq+pr+qr\right),$$
which holds even if $p+q \neq r$. In the case $p+q=r$, the above identity simplifies to
$$p^3+q^3-r^3+3pqr=0 \implies pq=\frac{r^3-p^3-q^3}{3r} \in \mathbb{Q}.$$
We assumed $p+q$ to be nonzero and rational, and now that pq is also rational, p and q must be the roots of some quadratic with rational coefficients. Specifically, p and q must follow the form $\frac{r \pm \sqrt{s}}{2}$ for some $s \in \mathbb{Q}_+$. Then,
$$(2p)^3=r^3 \pm 3r^2\sqrt{s}+3rs \pm s\sqrt{s}=r^3+3rs \pm \sqrt{s}\left(3r^2+s\right) \in \mathbb{Q},$$
implying $\sqrt{s}$ must be rational. But then p and q must be rational, contradicting the assumption that p and q are irrational. □
By direct substitution,
$$a+b+3\sqrt[3]{ab}\left(\sqrt[3]{a}+\sqrt[3]{b}\right)+6\sqrt[3]{a}+6\sqrt[3]{b}-2=0.$$
Because $x=\sqrt[3]{a}+\sqrt[3]{b}$ is irrational, the only way for $\left(3\sqrt[3]{ab}+6\right)x$ to be an integer is if $3\sqrt[3]{ab}+6=0$. This means $ab=-8$ and
$$x^3+6x-2=a+b+(0)-2=0 \implies a+b=2,$$
yielding the obvious solution $a=4$ and $b=-2$ and the answer $100a+b=\boxed{398}$.

10. The equation above could be factored into $(x-1)(x^{59}+x^{58}+\cdots+x+1-kx^{27}(x^2+x+1)) \geq 0$. The function is continuous and $(x-1)>0$ for $x>1$. So for $x=1$, $(x^{59}+x^{58}+\cdots+x+1-kx^{27}(x^2+x+1))=60-3k \geq 0$. Thus $k \geq 20$. Furthermore, when $a,b>0$, $a+b \geq 2\sqrt{ab}$. So, for $x>1$, $x^{59-n}+x^n \geq 2x^{29.5} \geq 2x^a$ where $a \leq 29.5$. Thus, if $k=20$, $(x^{59}+x^{58}+\cdots+x+1-20x^{27}(x^2+x+1)) \geq 60x^{29.5}-20x^{27}(x^2+x+1) \geq 0$, showing that the inequality holds for all $x>1$ when $k=\boxed{20}$.

11. Note that
$$a_{n+1}=-\frac{(a_n+1)+(a_n-1)}{(a_n-1)(a_n+1)}=\frac{2a_n}{1-a_n^2}.$$
The key observation is that the resulting expression for a_{n+1} resembles the tangent double-angle identity. If we choose θ so that $a_n=\tan\theta$, then $a_{n+1}=\tan 2\theta$. By induction, if we let $a_1=\tan\theta_1$, then $a_n=\tan\left(2^{n-1}\theta_1\right)$. The equality $a_6=a_1$ therefore occurs when $\tan\theta_1=\tan\left(2^{n-1}\theta_1\right) \implies 2^5\theta_1 \equiv \theta_1 \mod \pi \implies 31\theta_1 \equiv 0 \mod \pi$. Taking $\theta_1=\frac{k\pi}{31}$ for $k \in \{0,1,\ldots,30\}$ yields 31 distinct values of a_1. Because the tangent function has period π, no other values of θ_1 will generate new values of $\tan\theta_1$ other than the 31 values we already have. Thus, there are $\boxed{31}$ possible values for a_1.

12. The key observation is that the operation $\times$ mimics addition of fractions: $\frac{b}{a}+\frac{d}{c}=\frac{ad+bc}{ac}$. Therefore, $\times$ is commutative and associative, and
$$\frac{b_n}{a_n}=\sum_{k=2}^{n}\frac{g(k)}{f(k)}=\sum_{k=2}^{n}\frac{(k+1)^3-(k-1)^3}{k^2-1}=\sum_{k=2}^{n}\left(6+4\left(\frac{1}{k-1}-\frac{1}{k+1}\right)\right).$$
The sum $\sum_{k=2}^{n}\left(\frac{1}{k-1}-\frac{1}{k+1}\right)$ telescopes to $1+\frac{1}{2}-\frac{1}{n}-\frac{1}{n+1}$, so
$$\frac{b_n}{a_n}=6(n-1)+4\left(1+\frac{1}{2}-\frac{1}{n}-\frac{1}{n+1}\right)=6n-\frac{4(2n+1)}{n(n+1)} \in (6n-1,6n) \text{ for } n \geq 8.$$
Because 2020 is larger than 8 and not a multiple of 6, the answer is $n=\lceil 2020/6 \rceil=2022/6=\boxed{337}$.