Solutions

1. Since rotations don't matter, it doesn't matter where the first person sits. Without loss of generality, assume this person is a man. So the person next to him must be a women, of which there are 4 to choose from. The next seat is a man, of which there are 3 to choose from. So the total number is $4! \cdot 3! = 24 \cdot 6 = \boxed{144}$.

2. We first calculate the probability where Alexa rolled 3 and had 2 heads then divide that by the probability where Alexa has 2 heads. We first see that the probability that Alexa rolled 3 and had 2 heads is: $\frac{1}{6} \times \binom{3}{2} \times (\frac{1}{2})^3 = \frac{1}{16}$ With similar method, we find that the probability that Alexa has 2 heads is $\frac{33}{128}$. Thus probability that alexa rolled 3 becomes $\frac{1}{16} \times \frac{128}{33} = \frac{8}{33}$. Thus, the answer is $\boxed{41}$.

3. Notice that $D(2019) = 673$. In order for $D(n) > D(2019)$, we simply require n to be even and $n > 1346$. There are 336 such integers, so the answer is $336/2019 \implies 112/673 = \boxed{785}$

4. This is called the Gambler's Ruin Problem. The probability is simply $\frac{\text{distance from start to goal}}{\text{total distance}}$ which in this case is $\frac{5}{8}$, so the answer is $\boxed{13}$.

5. Enumerate the days of the week 1 through 7, and let X_i be an indicator random variable for the ith day of the week such that $X_i = 1$ when at least one student chooses to meet the professor on day i and $X_i = 0$ otherwise. The probability no students meet the professor that day is $\left(\frac{6}{7}\right)^5$, so the expected value of X_i is $1 - \left(\frac{6}{7}\right)^5$. By linearity of expectation, the expected value of $X_1 + X_2 + \cdots + X_7$ is the sum of the expected values of X_1 through X_7. Thus, the expected number of meeting days is $7\left(1 - \left(\frac{6}{7}\right)^5\right) = \frac{9031}{2401}$, so $\boxed{11432}$ is our answer.

6. First, observe that this problem is equivalent to a random walk starting at $x = 7$ and we need to find the probability that it gets to $x = 0$ where we move right with 75 percent probability and left with 25 percent probability. We first solve for q_1, the absorption probability starting at $x = 1$. We obtain the quadratic $3x^2 - 4x + 1 = 0$. Solving for x over the interval $[0, 1]$ yields $x = 1/3$. to compute q_7, we have $(1/3)^7 = 1/3^7$, so the answer is $1 + 2187 = \boxed{2188}$.

7. Label the starting vertex v_1, the vertices of distance 1 away from the v_1 as v_2, v_4, v_6, the vertices of distance 2 away as vertices v_3, v_5, v_7 and finally the terminating vertex as v_8. We will also use this notation to denote the expected number of moves until v_i is reached. We obtain that $v_1 = 1 + \frac{1}{3}(v_2 + v_4 + v_6)$ and get similar equations for $v_2, ..., v_7$. Clearly $v_8 = 0$ since we have finished once v_8 is obtained. Note further that $v_2 = v_4 = v_6$ and $v_3 = v_5 = v_7$ since they are all the same distance from both the starting and ending vertices. We eventually obtain that $v_1 = \boxed{10}$.

8. Suppose instead that Samantha initializes $a_1 = n + 1$ for any $n \in \mathbb{N} \cup \{0\}$. Let P_n be the probability that, after setting $a_1 = n + 1$, Samantha generates subsequent terms that have an odd sum (this sum includes every term in the sequence except a_1). Evidently, $P_0 = 0$, $P_1 = 1$, and $P_2 = 1$. In general, after Samantha selects a value k for a_2, the problem reduces to the case for P_{k-1}, in which the next sequence value can range from 1 to $k - 1$. Specifically, the probability that $a_2 + a_3 + \cdots + 1$ is odd is $1 - P_{k-1}$ if k is odd and P_{k-1} if k is even (this is because odd values of a_2 switch the parity of the sum). Thus,

$$P_{2n+1} = \frac{1}{2n+1}\left((1 - P_{2n}) + P_{2n-1} + (1 - P_{2n-2}) + P_{2n-3} + \cdots + (1 - P_0)\right) \text{ and}$$

$$P_{2n+2} = \frac{1}{2n+2}\left(P_{2n+1} + (1 - P_{2n}) + P_{2n-1} + (1 - P_{2n-2}) + \cdots + (1 - P_0)\right).$$

We notice some interesting relationships:

$$P_{2n+2} = \frac{1}{2n+2}\left(P_{2n+1} + (2n+1)P_{2n+1}\right) = P_{2n+1}, \text{ so}$$

$$P_{2n+1} = \frac{1}{2n+1}\left((1-P_{2n-1}) + P_{2n-1} + (1-P_{2n-3}) + P_{2n-3} + \cdots + (1-P_1) + P_1 + (1-P_0)\right)$$
$$= \frac{1}{2n+1}\left(1 - P_0 + \sum_{k=1}^{n}((1-P_{2k-1}) + P_{2k-1})\right) = \frac{1}{2n+1}\left(1+\sum_{k=1}^{n}1\right) = \frac{n+1}{2n+1}.$$

Hence, $P_{2n+2} = P_{2n+1} = \frac{n+1}{2n+1}$. For the problem at hand, we care about $a_1 = 2019$, which flips the parity of the sequence sum. The chance of the whole sequence sum being odd is the chance that $\{a_2, a_3, \ldots, 1\}$ has an even sum, which is $1 - P_{2018} = 1 - \frac{1009}{2017} = \frac{1008}{2017}$, so the answer is $1008 + 2017 = \boxed{3025}$.

9. $\boxed{57}$. Check out the solution at `http://www.math.wayne.edu/~danf/talks/CF.pdf`.

10. The diagram for this problem is a bipartite graph, in which the vertices can be split into two sets A and B such that no edges connect any two vertices in the same set. We generalize the solution to a bipartite graph with $|A| = |B| = n$ in which every vertex in A is connected to every vertex in B (and thus every vertex in B is connected to every vertex in A). Let $A = \{A_1, A_2, \ldots, A_n\}$ and $B = \{B_1, B_2, \ldots, B_n\}$, where we arbitrarily label the vertices in each set with indices from 1 to n. To make a cycle of length m (which is called an m-cycle), we must create a path over m distinct vertices that ends where it starts. Because the graph is bipartite, at each step in the path, we alternate from a vertex in A to a vertex in B or from a vertex in B to a vertex in A. In order to start and end at the same set, the cycle length must be even. Every $2k$-cycle we count will have k vertices in A and k vertices in B. There are $\binom{n}{k}$ ways to select the k vertices from A and $\binom{n}{k}$ ways to select the k vertices from B. Because the cycle is a loop, it is independent of whichever vertex we "start" our path at, so we arbitrarily fix one vertex A_{i_1} to "begin" the cycle. The remaining $k-1$ vertices from A have $(k-1)!$ ways to be ordered in the path sequence, and the k vertices from B have $k!$ ways to be ordered. Our current count is $\binom{n}{k}^2(k-1)!k! = \frac{(n!)^2}{(k!)^2((n-k)!)^2} \cdot \frac{(k!)^2}{k} = \frac{(n!)^2}{k((n-k)!)^2}$. Observe that this count treats the paths $\left\{A_{i_1}, B_{i_2}, A_{i_3}, \ldots, A_{i_{2k-1}}, B_{i_{2k}}\right\}$ and $\left\{A_{i_1}, B_{i_{2k}}, A_{i_{2k-1}}, \ldots, A_{i_3}, B_{i_2}\right\}$ as distinct cycles. We know that, in reality, the cycle loops are undirected, so reversing the direction in which we traverse the path does not produce a new cycle. Thus, our current count is an exact double count for the number of $2k$-cycles, so the correct count is

$$\frac{(n!)^2}{2k((n-k)!)^2}.$$

The smallest loop we can have is a 4-cycle, from $k = 2$, and the largest loop we can have is a $2n$-cycle, from $k = n$. Therefore, the answer is

$$\sum_{k=2}^{n}\frac{(n!)^2}{2k((n-k)!)^2}; \qquad n = 4 \implies \sum_{k=2}^{n}\frac{(n!)^2}{2k((n-k)!)^2} = \boxed{204}.$$