Solutions

1. We have $\int_{20}^{19} dx = [x]_{20}^{19} = 19 - 20 = \boxed{-1}$.

2. Letting $L = \lim\limits_{x \to 0^+} (\cos x)^{\ln x}$, we have $\ln L = \lim\limits_{x \to 0^+} \ln x \ln \cos x = \lim\limits_{x \to 0^+} \frac{\ln \cos x}{1/\ln x}$. We apply L'Hopital's rule: $\ln L = \lim\limits_{x \to 0^+} \frac{-\tan x}{-1/(x \ln^2 x)} = \lim\limits_{x \to 0^+} x \tan x \ln^2 x$. For nonnegative integers n, let $f(n) = \lim\limits_{x \to 0^+} x \ln^n x$. Observe that $f(0) = 0$ and, for $n > 0$, $f(n) = \lim\limits_{x \to 0^+} \frac{\ln^n x}{1/x} = \frac{(n \ln^{n-1} x) \cdot (1/x)}{-1/x^2} = -n \lim\limits_{x \to 0^+} x \ln^{n-1} x = -nf(n-1)$. It follows that $f(n) = 0$ for all n, so $\ln L = \tan(0) \cdot f(2) = 0 \implies L = \boxed{1}$.

3. Let $f(x) = \sum\limits_{n=0}^{\infty} \frac{x^{n+3}}{(n+3) \cdot n!}$ so that $f'(x) = \sum\limits_{n=0}^{\infty} \frac{x^{n+2}}{n!} = x^2 \sum\limits_{n=0}^{\infty} \frac{x^n}{n!} = x^2 e^x$. Observe that $f(0) = 0$. Then,
$$\sum_{n=0}^{\infty} \frac{1}{(n+3) \cdot n!} = f(1) = f(0) + \int_0^1 t^2 e^t \, dt = 0 + \left[e^t \left(t^2 - 2t + 2\right)\right]_0^1 = e - 2 \approx 0.71828,$$
so the answer is $\lfloor 100(e-2) \rfloor = \boxed{71}$.

4. Note that
$$\sum_{n=2}^{\infty} \frac{3n^2 + 3n + 1}{(n^2 + n)^3}$$
can be rewritten as
$$\sum_{n=2}^{\infty} \frac{1}{n^3} - \sum_{n=2}^{\infty} \frac{1}{(n+1)^3} = \frac{1}{8},$$
so the answer is $8 + 1 = \boxed{9}$.

5. Let $u = e^x - 1$ and $du = e^x dx$ so that
$$4 \int_{\ln 3}^{\ln 5} \frac{e^{3x}}{e^{2x} - 2e^x + 1} \, dx = 4 \int_2^4 \frac{(u+1)^2}{u^2} \, du = 4 \int_2^4 \left(1 + \frac{2}{u} + \frac{1}{u^2}\right) du = 4 \left[u + 2 \ln |u| - \frac{1}{u}\right]_2^4$$
$$= 4 \left(4 - \frac{1}{4} - 2 + \frac{1}{2} + 2 (\ln 4 - \ln 2)\right) = 9 + 8 \ln 2.$$
The answer is therefore $9 + 8 = \boxed{17}$.

6. For positive integers n, we have $(2n)!! = \prod\limits_{k=1}^{n} 2k = 2^n \prod\limits_{k=1}^{n} k = 2^n \cdot n!$. Observe that $(2n)!! = 2^n \cdot n!$ also holds for $n = 0$. Thus,
$$\sum_{n=0}^{\infty} \frac{1}{(2n)!!} = \sum_{n=0}^{\infty} \frac{1}{2^n \cdot n!} = \sum_{n=0}^{\infty} \frac{(1/2)^n}{n!} = e^{1/2} = \sqrt[4]{e^2},$$
where we used the Maclaurin series $e^x = \sum\limits_{n=0}^{\infty} \frac{x^n}{n!}$. Because $2.7 < e < 2.8$, $7.29 < e^2 < 7.84$, so $q = \boxed{7}$.

7. Taking the natural logarithm of both sides of $x^y = y^x$ yields $x \ln y = y \ln x \implies \frac{\ln y}{y} = \frac{\ln x}{x}$. We use implicit differentiation:
$$\frac{\frac{1}{y} \cdot y - 1 \cdot \ln y}{y^2} \frac{dy}{dx} = \frac{\frac{1}{x} \cdot x - 1 \cdot \ln x}{x^2} \implies \frac{dy}{dx} = \left(\frac{y}{x}\right)^2 \frac{1 - \ln x}{1 - \ln y}.$$
When $x = 4$, we have $y = 2$ because $4^2 = 2^4$, so
$$f'(4) = \frac{1}{2^2} \cdot \frac{1 - \ln 4}{1 - \ln 2} = \frac{1 - 2 \ln 2}{4 - 4 \ln 2} = \frac{-1 + 2 - 2 \ln 2}{4 - 4 \ln 2} = \frac{1}{2} - \frac{1}{4 - \ln 16} \implies a + b + c = 2 + 4 + 16 = \boxed{22}.$$

8. Let a be the positive real number such that the circle of radius 4 is tangent to the curve of $y = x^2$ at the points $P\left(-a, a^2\right)$ and $Q\left(a, a^2\right)$, and let C be the center of the circle. The slope of the line tangent to $y = x^2$ at $x = a$ is $\frac{d}{dx}x^2\big|_{x=a} = 2a$, so the slope of $\overline{QC}$ is $-\frac{1}{2a}$ because $\overline{QC}$ is perpendicular to the tangent line. The y-coordinate of C is therefore $-\frac{1}{2a}(-a) = \frac{1}{2}$ larger than the y-coordinate of Q. Since $Q = \left(a, a^2\right)$, we conclude that $C = \left(0, a^2 + \frac{1}{2}\right)$. Let $R = \left(0, a^2\right)$. Note that $\triangle CQR$ is a right triangle with legs a and $\frac{1}{2}$ and a hypotenuse of 4 (the circle's radius). By the Pythagorean theorem, $a^2 = 4^2 - \frac{1}{2^2} = \frac{63}{4}$, so $\frac{p}{q} = \frac{63}{4} + \frac{1}{2} = \frac{65}{4}$, yielding $p + q = 65 + 4 = \boxed{69}$.

9. We generalize this problem to a cylinder of radius R and a string of length $R\pi$ with one end pinned at $(x, y) = (R, 0)$. Let S be the circular base of the cylinder. Clearly, the string can sweep out a semicircle to the right of the line $x = R$ with radius $R\pi$, whose area is $\frac{1}{2}R^2\pi^3$. The remaining area that the string can cover is swept out as the string wraps around S in either direction; the farthest the string can wrap is $(-R, 0)$, covering half the circumference of S, or $R\pi$, the full length of the string. Using the parameter $\theta \in (0, \pi]$, we let $(x(\theta), y(\theta))$ be the position of the free end of the string when it is wrapped around θ radians of S (in a counterclockwise direction) and the remainder of the string is taut and lies along the line tangent to S at $(R\cos\theta, R\sin\theta)$. If $R\theta$ is the length of the wrapped portion of the string, then $R(\pi - \theta)$ is the length of the straight portion. The slope of the straight portion is $-\frac{\cos\theta}{\sin\theta}$, so $(x(\theta), y(\theta)) = (R\cos\theta - R(\pi-\theta)\sin\theta, R\sin\theta + R(\pi - \theta)\cos\theta)$. The area above the x-axis bounded by the curve of all $(x(\theta), y(\theta))$ for $0 < \theta \le \pi$ is given by

$$A = \frac{1}{2}\int_0^\pi \sqrt{x^2(\theta) + y^2(\theta)}\left(\sqrt{x^2(\theta) + y^2(\theta)}\, d\theta\right)$$

$$= \frac{R^2}{2}\int_0^\pi \left(\cos^2\theta + (\pi-\theta)^2\sin^2\theta - 2(\pi-\theta)\cos\theta\sin\theta + \sin^2\theta + (\pi-\theta)^2\cos^2\theta + 2(\pi-\theta)\sin\theta\cos\theta\right) d\theta$$

$$= \frac{R^2}{2}\int_0^\pi \left(1 + (\pi-\theta)^2\right) d\theta = \frac{R^2}{2}\int_0^\pi \left(1 + u^2\right) du = \frac{R^2}{2}\left(\pi + \frac{\pi^3}{3}\right).$$

This area measure includes the area of the half of S that lies above the x-axis, which is $\frac{\pi}{2}R^2$. Thus, the string sweeps out an area of $A - \frac{\pi}{2}R^2 = \frac{R^2\pi^3}{6}$ above the x-axis and to the left of $x = R$. By symmetry, the same area is swept out below the x-axis and to the left of $x = R$. Adding these two areas to the initial semicircular area yields the total area,

$$2 \cdot \frac{R^2\pi^3}{6} + \frac{R^2\pi^3}{2} = \frac{5R^2\pi^3}{6}.$$

Taking $R = 6$ makes the area $30\pi^3$, giving $m = \boxed{30}$.

10. Because $AB = 2$, $AC + CB = 10$. A and B are fixed points, so the locus of points C such that $AC + CB = 10$ is an ellipse with foci A and B. Suppose that this ellipse has the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$; let $g(x, y) = \frac{x^2}{a^2} + \frac{y^2}{b^2}$. To maximize $f(x, y) = x + y$ over the curve $g(x, y) = 1$, we can use Lagrange Multipliers, which states that critical points $(x^\star, y^\star)$ of f over the curve $g(x, y) = 1$ satisfy

$$\nabla f\left(x^\star, y^\star\right) \propto \nabla g\left(x^\star, y^\star\right).$$

Since $\nabla f(x, y) = \langle 1, 1\rangle$ and $\nabla g(x, y) = \left\langle \frac{2x}{a^2}, \frac{2y}{b^2}\right\rangle$, we have $\frac{x^\star}{a^2} = \frac{y^\star}{b^2} \implies x^\star = \frac{y^\star a^2}{b^2} \implies 1 = y^{\star 2}\left(\frac{a^2}{b^4} + \frac{1}{b^2}\right) \implies y^\star = \pm\frac{b^2}{\sqrt{a^2+b^2}}$ and $x^\star = \pm\frac{a^2}{\sqrt{a^2+b^2}}$. Clearly, taking both $x^\star$ and $y^\star$ to be positive will make the extremum of f at $(x^\star, y^\star)$ a maximum, so the maximum possible value of $x + y$ is

$$\frac{a^2 + b^2}{\sqrt{a^2 + b^2}} = \sqrt{a^2 + b^2}.$$

Note that the points $(a, 0)$ and $(0, b)$ lie on the ellipse of interest. In our case, this means that $(a-1)+(a+1) = 10$ and $2\sqrt{1 + b^2} = 10$, so we get $a = 5$, $b = \sqrt{24}$, and $\max(x+y) = \sqrt{5^2 + \sqrt{24}^2} = \boxed{7}$.