Johns Hopkins Math Tournament 2019

Individual Round: Algebra

February 9, 2019

Instructions

- **DO NOT TURN OVER THIS PAPER UNTIL TOLD TO DO SO.**
- This test contains 10 questions to be solved individually in 60 minutes.
- All answers will be integers.
- Only answers written on the appropriate area on the answer sheet will be considered for grading.
- Problems are weighted relative to their difficulty, determined by the number of students who solve each problem.
- No translators, books, notes, slide rules, calculators, abaci, or other computational aids are permitted. Similarly, graph paper, rulers, protractors, compasses, and other drawing aids are not permitted.
- If you believe the test contains an error, immediately tell your proctor and if necessary, report the error to the front desk after the end of your exam.
- Good luck!

1. The sum of the squares of two numbers is 2019 and the product of the two numbers is 95. What is the sum of the two numbers?

2. How many ordered pairs of integers (a, b) exist such that $1/a + 1/b = 1/4$?

3. How many natural numbers less than 2019 are there such that its remainder when divided by 2 is 1, when divided by 3 is 2, when divided by 4 is 3, and when divided by 5 is 4?

4. Let $P(x)$ be a polynomial with real coefficients where the sum of the coefficients is equal to 2019. Also, $P(x)$ satisfies
$$P(-x) = -P(x)$$
The remainder, $Q(x)$, obtained by dividing $P(x)$ by $x^3 - x$ has the form $px^2 + qx + r$, where p, q, and r are constants. Find $p + q + r$.

5. The given polynomial $P(X)$ has leading coefficient 1 and satisfies the functional equation below:
$$(X + 1)P(X) = (X - 10)P(X + 1)$$
Compute P(5).

6. $\binom{1000}{0} - \binom{1000}{2} + \binom{1000}{4} - \cdots + \binom{1000}{1000} = 2^A$. Find A.

7. Given the quadratic equation $ax^2 - bx + c = 0$, where $a, b, c \in \mathbb{R}$, find the coefficients a, b, c such that the equation has the roots a, b and discriminant c. Compute $\frac{4c}{ab}$.

8. The equation below has only one real solution of the form a/b where a and b are coprime. Find $a + b$.
$$x^3 + (x + 1)^3 + (x + 2)^3 + (x + 3)^3 = 0$$

9. Given that the equation $(1 + x +^2 +x^3 + ... + x^{17})^2 - x^{17} = 0$ has 34 complex roots of the form $z_k = r_k[\cos(2\pi a_k) + i\sin(2\pi a_k)]$, $k = 1, 2, 3, ..., 34$, with $0 < a_1 \leq a_2 \leq a_3 \leq ... \leq a_{34} < 1$ and $r_k > 0$. Find $a_1 + a_2 + a_3 + a_4 + a_5$. Given the answer is in the form $\frac{\alpha}{\beta}$, compute $\alpha + \beta$.

10. The roots, a, b, c, of the equation $x^3 - 4x^2 + 5x - 19/10 = 0$ are real and can form the sides of a triangle. Given the area of the triangle has form $\sqrt{q}/p$ where p is an integer and $\sqrt{q}$ is in simplest radical form, find $p + q$.