# Ross Program 2020 Application Problems

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institute 34 This document is part of the application to the Ross Mathematics Program, and will remain posted at https://rossprogram.org/students/to-apply from January through March.

> The Admission Committee will make acceptance decisions on a rolling basis, starting in March 2020. The deadline for applications is April 1, but spaces will fill as applications arrive. For adequate consideration of your application, it is best to **submit** your solutions well before the end of March.

Work independently on the problems below. We are interested in seeing how you approach unfamiliar math problems, not whether you can find answers by searching through web sites or books, or by asking other people.

#### Please submit your own work on these problems.

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institute # For each problem, explore the situation (with calculations, tables, pictures, etc.), observe patterns, make some guesses, test the truth of those guesses, and write logical proofs when possible. Where were you led by your experimenting?

a complete solution. If you've seen one of the problems before (e.g. in a class or online), please include a reference along with recurrent. Include your thoughts (but not your scratch-paper) even if you might not have found

We are not looking for quick answers written in minimal space. Instead, we hope to see evidence of your explorations, conjectures, and proofs written in a readable format. Divitute ## # 18

The quality of mathematical exposition, as well as the correctness and completeness of your solutions, are factors in admission decisions.

Please convert your problem solutions into a PDF file. You may type the solutions using LATEX or a word processor, and convert the output to PDF format. Alternatively, you may scan your solutions from a handwritten paper copy, and convert the output to PDF. (Please use dark pencil or pen and write on only one side of the paper.) Submitting photos of your work is possible but not recommended: The resulting PDF files are often large, and the writing can be blurry and difficult to read.

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withte the the the Note: Unlike the problems here, each Ross Program course concentrates deeply on one subject. These problems are intended to assess your general mathematical background and interests.

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**Definition.** Sequence  $A = (a_n)$  is an "alt-basis" if every positive integer is uniquely A-expressible. That is, for every integer m > 0, there is exactly one way to express m as an alternating sum of a finite subsequence of A.

**Examples.** Sequence  $B = (2^{n-1}) = (1, 2, 4, 8, 16, ...)$  is not an alt-basis because some numbers are B-expressible in more than one way. For instance 3 = -1 + 4 =1 - 2 + 4.

Sequence  $C = (3^{n-1}) = (1, 3, 9, 27, 81, ...)$  is not an alt-basis because some numbers (like 4 and 5) are not C-expressible. 而时间他都林道像

(a) Let  $D = (2^n - 1) = (1, 3, 7, 15, 31, ...)$ . Note that: 1 = 1, 2 = -1 + 3, 3 - 3 - 4 $1 = 1, 2 = -1 + 3, 3 = 3, 4 = -3 + 7, 5 = 1 - 3 + 7, 6 = -1 + 7, 7 = 7, 8 = -7 + 15, 9 = 1 - 7 + 15, \dots$ 

Prove that D is an alt-basis.

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(b) Can some E = (2, 3, ...) be an alt-basis? That is, can you construct an alt-basis  $E = (e_n)$  with  $e_1 = 2$  and  $e_2 = 3$ ? institute # matitute #

(c) Can some F = (1, 4, ...) be an alt-basis? Justify your answer.

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whether a given sequence  $A = (a_n)$  is an alt-basis? (d) Investigate some other examples. Is there some fairly simple test to determine 面动机能称林塔张 astitute the the " is the The stitute the second institute ##

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titute 新林·诺陀 titute # # # B time # # '3 1% itute mat 's PR institute 30 A polynomial f(x) has the factor-square property (or FSP) if f(x) is a factor of  $f(x^2)$ . For instance, g(x) = x - 1 and h(x) = x have FSP, but k(x) = x + 2 does not.

Reason: x - 1 is a factor of  $x^2 - 1$ , and x is a factor of  $x^2$ , but x + 2 is not a factor of  $x^2 + 2$ .

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Multiplying by a nonzero constant "preserves" FSP, so we restrict attention to polynomials that are *monic* (i.e., have 1 as highest-degree coefficient).

What patterns do monic FSP polynomials satisfy? To make progress on this topic, investigate the following questions and justify your answers.

(a) Are x and x - 1 the only monic FSP polynomials of degree 1? (b) List all the maximum POP

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- mytute # # B To start, note that  $x^2$ ,  $x^2 - 1$ ,  $x^2 - x$ , and  $x^2 + x + 1$  are on that list. Some of them are products of FSP polynomials of smaller degree. For instance, ·3 92  $x^2$  and  $x^2 - x$  arise from degree 1 cases. However,  $x^2 - 1$  and  $x^2 + x + 1$  are new, 面动曲称茶塔 stitute ## not expressible as a product of two smaller FSP polynomials. Which terms in your list of degree 2 examples are new?
  - (c) List all the monic FSP polynomials of degree 3. Which of those are new?

 (d) Answers to the previous questions may depend on what coefficients are allowed.
List the monic FSP polynomials of degree 3 that have integer and separately list these GF. integers.

Can you make similar lists for degree 4?

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are not all integers? matinue ## # ' K Are there examples of monic FSP polynomials with real number coefficients that myitute # # 'S Turitute # # 3 within the the the 's

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### 拉加伦斯林·法家 titute mark 'S The titute # # '3 1% little math & M Ro Problem 3 institute # Here we work in the system of integer polynomials. Those are polynomials of the form $f(x) = r_n x^n + \cdots + r_1 x + r_0$ where every coefficient $r_i$ is an integer. General question: multille # # 3 PS That is, when do there exist integer polynomials ax + b and cx + d exists integer polynomials P(x) and Q(x) with $P(x) \cdot (ax + b) + Q(x) \cdot (cx + b) = 1.2$ Ro When does some combination of the polynomials ax + b and cx + d equal 1? We concentrate here on cases when c = 0. That is, no integer polynomials P(x), Q(x) can satisfy: $P(x) \cdot (2x+5) + Q(x)$ 而时间的新祥等除 mutute # # 3 PS Ro (b) Find a combination of 2x + 5 and 4 that equals 1. (d) Investiger How about 15x + 9 and 20? (c) Does some combination of 15x + 9 and 25 equal 1? 冰陽隊 R (d) Investigate further examples of ax + b and d, deciding in each case whether 1 is a combination. What patterns do you detect? Can you prove that some of your observed patterns always hold true? mutute # # # B 而如此他教林塔然 而如此他教林後然 而如此他教林後然 而此此他就林塔然 matinte # # 13 PR Y. Inditute # # '& R 而如此他就林塔路 而如此他教林後然 mutute ## # '& R 而时间很新林塔路 而如此他教林後然 Y. Inditute # # '& R 而如此他就林塔路 mutute ## # '& R 面的机机都林塔像 mutute ## # '& R 而时间很新林塔路 No. Y.

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