New *Wilker-type Inequalities for Trigonometric Functions*

Junyang Chen, Yue Pan , Shuyang Zhang

Abstract

In this paper, we established two new *Wilker-Type inequalities for* trigonometric functions and proved the validity of such inequalities . *We have also given a concise proof of conventional Wilker's inequality and of* Hungens-type inequality.

Key words: *Wilker's inequalities,* Hungens-type inequality, *trigonometric Functions, power series expansion*

1 Introduction

In 1989, J.B.Wilker[2]proposed two open questions in the American Mathematical Monthly, among which the first one was:

Problem 1. If $0 < x < \frac{\pi}{2}$,then

$$(\frac{\sin x}{x})^2 + \frac{\tan x}{x} > 2 \tag{1.1}$$

the second one was:

Problem 2. If $0 < x < \frac{\pi}{2}$,find the largest constant c such that

$$(\frac{\sin x}{x})^2 + \frac{\tan x}{x} > 2 + cx^3 \tan x \tag{1.2}$$

They have aroused remarkable interest of many mathematicians who conducted a huge number of researches upon this topic.

J.S. Sumner et al.[3] proved that the truthfulness of (1.1) and (1.2) resulted in the following theorem 1:

Theorem 1. If $0 < x < \frac{\pi}{2}$,then

$$\frac{16}{\pi^4} x^3 \tan x < (\frac{\sin x}{x})^2 + \frac{\tan x}{x} - 2 < \frac{8}{45} x^3 \tan x \qquad (1.3)$$

Furthermore, $\frac{16}{\pi^4}$ and $\frac{8}{45}$ are the best constants in(1.3).

Recently, Zhu[6] gave a new simple proof of inequalities(1.1), and Zhang and Zhu[4]gave a new *elementary proof of Wilker's inequalities*(1.3).Zhu[5] showed *some new Wilker-Type inequalities for circular and hyperbolic functions.* L.Zhu and Marija Nenezić[11]gave new approximation inequalities for circular functions.

Another inequality, the Huygens inequality [13], aroused our interest in the process of researching. Such an inequality asserts that

If $0 < x < \frac{\pi}{2}$, then

$$2(\frac{\sin x}{x}) + \frac{\tan x}{x} > 3 \qquad (1.4)$$

In recent years, lots of papers concerning Huegens inequality has arisen, including but not limited to Zhu's[15], in which he has shown some new inequalities of the Huygens-type for trigonometric and hyperbolic functions; Chen's[16], in which he has given some new inequalities of the Huygens-type for inverse trigonometric and inverse hyperbolic functions; and also Chen and Cheung's,[14] in which they have shown but have failed to demonstrate an exact proof of Wilker and Huygens type inequalities including the following

Theorem 2. If $0 < x < \frac{\pi}{2}$,then

$$\frac{3}{20} x^3 \tan x < 2(\frac{\sin x}{x}) + \frac{\tan x}{x} - 3 < \frac{16}{\pi^4} x^3 \tan x \qquad (1.5)$$

Furthermore, $\frac{16}{\pi^4}$ and $\frac{3}{20}$ are the best constants in (1.5).

Subsequently, we establish two new *Wilker-Type inequalities theorem* 3 *and theorem* 4----*the main results of this paper. We'll show a concise proof of Wilker's inequality* (1.3) along with a proof of (1.5) using similarly succinct methods.

2 Some Lemmas

Lemma 1 (see [12], P.20, P.23). For $n \geq 1$,we $(-1)^{n-1}B_{2n} > 0$, have where B_n $(n \in N)$ are a type of numbers called the Bernoulli Numbers, defined by the following formula :

$$\frac{te^t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}$$

Lemma 2(see [7-11]) let B_{2n} be the even-indexed Bernoulli numbers, $n \geq 1, n \in N$ then

$$\frac{2^{2n-1}-1}{2^{2n+1}-1}\frac{(2n+2)(2n+1)}{\pi^2} < \frac{|B_{2n+2}|}{|B_{2n}|} < \frac{2^{2n}-1}{2^{2n+2}-1}\frac{(2n+2)(2n+1)}{\pi^2}$$

Lemma 3(see [12], P.23,[5]). We know that the power expansions of tangent function and cotangent function are the following

$$\tan x = \sum_{n=1}^{\infty} (2^{2n}-1)2^{2n}|B_{2n}|\frac{x^{2n-1}}{(2n)!} \qquad |x| < \frac{\pi}{2} \tag{2.1}$$

$$\cot x = \frac{1}{x} - \sum_{n=1}^{\infty} 2^{2n}|B_{2n}|\frac{x^{2n-1}}{(2n)!} \qquad 0 < |x| < \pi \tag{2.2}$$

So, we can get the power expansions for the following functions

$$\sec^2 x = (\tan x)' = \sum_{n=1}^{\infty} (2^{2n}-1)2^{2n}(2n-1)|B_{2n}|\frac{x^{2n-2}}{(2n)!} \qquad |x| < \frac{\pi}{2} \tag{2.3}$$

$$\csc^2 x = (-\cot x)' = \frac{1}{x^2} + \sum_{n=1}^{\infty} 2^{2n}(2n-1)|B_{2n}|\frac{x^{2n-1}}{(2n)!} \quad 0<|x|<\pi \tag{2.4}$$

$$\cot^2 x = \csc^2 x - 1 = \frac{1}{x^2} + \sum_{n=1}^{\infty} 2^{2n}(2n-1)|B_{2n}|\frac{x^{2n-1}}{(2n)!} - 1 \quad 0<|x|<\pi \tag{2.5}$$

$$\csc 2x = \frac{1}{2x} + \frac{1}{2}\sum_{n=1}^{\infty} 2^{2n}(2^{2n}-2)|B_{2n}|\frac{x^{2n-1}}{(2n)!} \qquad 0<|x|<\frac{\pi}{2} \tag{2.6}$$

The formula (2.6) holds true because of the existence of the equation as follows:

$$\csc 2x = \frac{1}{\sin 2x} = \frac{1}{2\sin x\cos x} = \frac{\sin^2 x + \cos^2 x}{2\sin x\cos x} = \frac{1}{2}(\tan x + \cot x)$$

□

$$\frac{\sin x}{\cos^3 x} = \frac{1}{2}(\sec^2 x)' = \frac{1}{2}\sum_{n=2}^{\infty}(2^{2n}-1)2^{2n}(2n-1)(2n-2)|B_{2n}|\frac{x^{2n-3}}{(2n)!} \qquad |x|<\frac{\pi}{2} \tag{2.7}$$

$$\cot x\csc^2 x = -\frac{1}{2}(\cot^2 x)' = \frac{1}{x^3} - \frac{1}{2}\sum_{n=2}^{\infty} 2^{2n}(2n-1)(2n-2)|B_{2n}|\frac{x^{2n-3}}{(2n)!} \quad 0<|x|<\pi \tag{2.8}$$

$$\csc x = \frac{1}{x} + \sum_{n=1}^{\infty}(2^{2n}-2)|B_{2n}|\frac{x^{2n-1}}{(2n)!} \qquad 0<|x|<\frac{\pi}{2} \tag{2.9}$$

$$\cot x\csc x = (-\csc x)' = \frac{1}{x^2} - \sum_{n=1}^{\infty}(2n-1)(2^{2n}-2)|B_{2n}|\frac{x^{2n-2}}{(2n)!} \quad 0<|x|<\frac{\pi}{2} \tag{2.10}$$

$$\begin{aligned}\frac{1}{\sin^3 x} &= \frac{1}{x^3} + \frac{1}{2}\sum_{n=2}^{\infty}(2^{2n}-2)(2n-1)(2n-2)|B_{2n}|\frac{x^{2n-3}}{(2n)!} \\ &\quad + \frac{1}{2x} + \frac{1}{2}\sum_{n=1}^{\infty}(2^{2n}-2)|B_{2n}|\frac{x^{2n-1}}{(2n)!} \qquad 0<|x|<\frac{\pi}{2}\end{aligned} \tag{2.11}$$

The formula (2.11) holds true because of the existence of the equation as follows:

$$\frac{1}{\sin^3 x} = \frac{1}{2}(-\csc x\cot x)' + \frac{1}{2}\csc x = \frac{1}{2}(\csc x)'' + \frac{1}{2}\csc x$$

□

3 main results of this paper

Theorem 3. If $0 < x < \frac{\pi}{2}$,then

$$(\frac{\sin x}{x})^2 + \frac{\tan x}{x} - 2 > \frac{8}{45} x^4 \left(\frac{\tan x}{x}\right)^{\frac{6}{7}} \tag{3.1}$$

Holds true. Furthermore, $\frac{8}{45}$ is the best constant in (3.1).

Proof . Let $f(x) = \dfrac{\sin^2 x + x\tan x - 2x^2}{x^6 (\frac{\tan x}{x})^{\frac{6}{7}}}$ then

$$f'(x) = \frac{(\sin 2x + \tan x + x\sec^2 x - 4x)x^6(\frac{\tan x}{x})^{\frac{6}{7}} - (\sin^2 x + x\tan x - 2x^2)[x^6(\frac{\tan x}{x})^{\frac{6}{7}}]'}{x^{12}(\frac{\tan x}{x})^{\frac{12}{7}}}$$

where

$$[x^6(\frac{\tan x}{x})^{\frac{6}{7}}]' = 6x^5(\frac{\tan x}{x})^{\frac{6}{7}} + \frac{6}{7}x^6(\frac{\tan x}{x})^{-\frac{1}{7}}\frac{x\sec^2 x - \tan x}{x^2}$$

thus

$$f'(x) = \frac{7x\tan x(\sin 2x + \tan x + x\sec^2 x - 4x) - (\sin^2 x + x\tan x - 2x^2)(36\tan x + 6x\sec^2 x)}{7x^8(\frac{\tan x}{x})^{\frac{13}{7}}}$$

$$= \frac{g(x)}{7x^8(\frac{\tan x}{x})^{\frac{13}{7}}}$$

where

$$g(x) = 7x\tan x(\sin 2x + \tan x + x\sec^2 x - 4x) - (\sin^2 x + x\tan x - 2x^2)(36\tan x + 6x\sec^2 x)$$

$$= 14x\sin^2 x - 36\sin^2 x\tan x + 12x^3\sec^2 x - 35x\tan^2 x + x^2\sec^2 x\tan x + 44x^2\tan x$$

$$= \sin^2 x(14x - 36\tan x + 12x^3\frac{1}{\sin^2 x\cos^2 x} - 35x\sec^2 x + x^2\frac{1}{\sin x\cos^3 x} + 44x^2\frac{1}{\sin x\cos x})$$

$$=\sin^2 x(14x-36\tan x+12x^3\frac{\sin^2 x+\cos^2 x}{\sin^2 x\cos^2 x}-35x\sec^2 x+x^2\frac{\sin^2 x+\cos^2 x}{\sin x\cos^3 x}+88x^2\csc 2x)$$

$$=\sin^2 x(14x-36\tan x+12x^3\sec^2 x+12x^3\csc^2 x-35x\sec^2 x+x^2\frac{\sin x}{\cos^3 x}+90x^2\csc 2x)$$

$$=\sin^2 x\cdot s(x)$$

where

$$s(x)=14x-36\tan x+12x^3\sec^2 x+12x^3\csc^2 x-35x\sec^2 x+x^2\frac{\sin x}{\cos^3 x}+90x^2\csc 2x$$

By using (2.1)(2.3)(2.4)(2.6)(2.7),we can obtain

$$s(x)=14x-36\sum_{n=1}^{\infty}2^{2n}(2^{2n}-1)|B_{2n}|\frac{x^{2n-1}}{(2n)!}+12x^3\sum_{n=1}^{\infty}2^{2n}(2^{2n}-1)(2n-1)|B_{2n}|\frac{x^{2n-2}}{(2n)!}$$

$$+12x^3(\frac{1}{x^2})+12x^3\sum_{n=1}^{\infty}2^{2n}(2n-1)|B_{2n}|\frac{x^{2n-2}}{(2n)!}+\frac{x^2}{2}\sum_{n=2}^{\infty}2^{2n}(2^{2n}-1)(2n-1)(2n-2)|B_{2n}|\frac{x^{2n-3}}{(2n)!}$$

$$-35x\sum_{n=1}^{\infty}2^{2n}(2^{2n}-1)(2n-1)|B_{2n}|\frac{x^{2n-2}}{(2n)!}+90x^2(\frac{1}{2x})+45x^2\sum_{n=1}^{\infty}2^{2n}(2^{2n}-2)|B_{2n}|\frac{x^{2n-1}}{(2n)!}$$

$$=71x+\sum_{n=1}^{\infty}(24n2^{2n}+33\cdot 2^{2n}-90)2^{2n}|B_{2n}|\frac{x^{2n+1}}{(2n)!}+\sum_{n=2}^{\infty}2^{2n}(2^{2n}-1)(2n-1)(n-1)|B_{2n}|\frac{x^{2n-1}}{(2n)!}$$

$$-\sum_{n=1}^{\infty}(70n+1)2^{2n}(2^{2n}-1)|B_{2n}|\frac{x^{2n-1}}{(2n)!}$$

$$=71x+\sum_{n=1}^{\infty}(24n2^{2n}+33\cdot 2^{2n}-90)2^{2n}|B_{2n}|\frac{x^{2n+1}}{(2n)!}+\sum_{n=2}^{\infty}2^{2n}(2^{2n}-1)(2n-1)(n-1)|B_{2n}|\frac{x^{2n-1}}{(2n)!}$$

$$-71x-\sum_{n=2}^{\infty}(70n+1)2^{2n}(2^{2n}-1)|B_{2n}|\frac{x^{2n-1}}{(2n)!}$$

$$=\sum_{n=1}^{\infty}(24n2^{2n}+33\cdot 2^{2n}-90)2^{2n}|B_{2n}|\frac{x^{2n+1}}{(2n)!}+\sum_{n=2}^{\infty}(2n^2-73n)2^{2n}(2^{2n}-1)|B_{2n}|\frac{x^{2n-1}}{(2n)!}$$

$$=\sum_{n=2}^{\infty}[24(n-1)2^{2n-2}+33\cdot 2^{2n-2}-90)2^{2n-2}|B_{2n-2}|\frac{x^{2n-1}}{(2n-2)!}+\sum_{n=2}^{\infty}(2n^2-73n)2^{2n}(2^{2n}-1)|B_{2n}|\frac{x^{2n-1}}{(2n)!}$$

$$=\sum_{n=2}^{\infty}[(24n2^{2n-2}+9\cdot 2^{2n-2}-90)(2n)(2n-1)\frac{|B_{2n-2}|}{|B_{2n}|}+4(2n^2-73n)(2^{2n}-1)]2^{2n-2}|B_{2n}|\frac{x^{2n-1}}{(2n)!}$$

$$=\sum_{n=2}^{\infty}a_n 2^{2n-2}|B_{2n}|\frac{x^{2n-1}}{(2n)!}=\sum_{n=3}^{\infty}a_n 2^{2n-2}|B_{2n}|\frac{x^{2n-1}}{(2n)!}$$ (for $a_2=0, when \quad n=2$)

Where, $a_n=(24n2^{2n-2}+9\cdot 2^{2n-2}-90)(2n)(2n-1)\frac{|B_{2n-2}|}{|B_{2n}|}+4(2n^2-73n)(2^{2n}-1)$

Theorem 2 shall be correct if we can successfully prove the following inequality:

$a_n>0, when \quad n\geq 3$

According to lemma 2, we have

$$\frac{|B_{2n-2}|}{|B_{2n}|}>\frac{2^{2n}-1}{2^{2n-2}-1}\cdot\frac{\pi^2}{(2n)(2n-1)}$$

So

$$a_n>(24n2^{2n-2}+9\cdot 2^{2n-2}-90)(2n)(2n-1)\frac{2^{2n}-1}{2^{2n-2}-1}\cdot\frac{\pi^2}{(2n)(2n-1)}+4(2n^2-73n)(2^{2n}-1)$$

$$=(24n2^{2n-2}+9\cdot 2^{2n-2}-90)\frac{2^{2n}-1}{2^{2n-2}-1}\cdot\pi^2+4(2n^2-73n)(2^{2n}-1)$$

$$=\frac{2^{2n}-1}{2^{2n-2}-1}[24n\pi^2 2^{2n-2}+9\pi^2\cdot 2^{2n-2}-90\pi^2+4(2n^2-73n)(2^{2n-2}-1)]$$

$=\frac{2^{2n}-1}{2^{2n-2}-1}b_n$, where,

$$b_n=[24n\pi^2 2^{2n-2}+9\pi^2\cdot 2^{2n-2}-90\pi^2+4(2n^2-73n)(2^{2n-2}-1)]$$

$$=24n\pi^2 2^{2n-2}+9\pi^2\cdot 2^{2n-2}+8n^2\cdot 2^{2n-2}+282n-282n\cdot 2^{2n-2}-8n^2-90\pi^2$$

$$b_3=11890-11610>0$$

$$b_n=(24n\pi^2+8\pi^2+8n^2-282n)2^{2n-2}+(282n-90\pi^2)+(\pi^2\cdot 2^{2n-2}-8n^2)$$

$=c_n 2^{2n-2}+(282n-90\pi^2)+(\pi^2\cdot 2^{2n-2}-8n^2)$,

where

$$c_n=24n\pi^2+8\pi^2+8n^2-282n$$

When $\quad n>3$, $(282n-90\pi^2)>0, (\pi^2\cdot 2^{2n-2}-8n^2)>0$

And when $n \geq 9$ $c_n = 24n\pi^2 + 8\pi^2 + 8n^2 - 282n > 216n + 8n^2 - 282n = (8n - 66)n > 0$

We can easily obtain

$c_4 = 104\pi^2 - 1000 \approx 1025 - 1000 > 0$

$c_5 = 128\pi^2 - 1210 \approx 1262 - 1210 > 0$

$c_6 = 152\pi^2 - 1404 \approx 1498 - 1404 > 0$

$c_7 = 176\pi^2 - 1582 \approx 1735 - 1582 > 0$

$c_8 = 300\pi^2 - 512 > 2700 - 512 > 0$

So $b_n > 0$ when $n \geq 3$,of cause, $a_n > 0$ when $n \geq 3$.

As we can see ,all coefficients of the polynomial $s(x)$ are positive integers.

When $x > 0, s(x) > 0$, $g(x) = \sin^2 x \cdot s(x) > 0$.

when $x > 0,\ f'(x) = \dfrac{g(x)}{7x^8 (\frac{\tan x}{x})^{\frac{13}{7}}} > 0$

we can conclude that *f*(*x*) is strictly increasing on $(0, \frac{\pi}{2})$

$so \quad f(x) > \lim_{x \to 0^+} f(x)$

Furthermore $\lim_{x \to 0^+} f(x) = \frac{8}{45}$,and the proof of Theorem3 is complete.

Theorem 4. If $0 < x < \frac{\pi}{2}$,then

$$\left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x} - 2 > \frac{2}{45} x^4 \left(\frac{\sin x}{x}\right)^{\frac{6}{7}} \qquad (3.2)$$

Holds. Furthermore, $\frac{2}{45}$ is the best constant in (3.2).

Proof.

Let

$$f(x)=\frac{x^2+\frac{x}{2}\sin 2x-2\sin^2 x}{x^6\left(\frac{\sin x}{x}\right)^{\frac{20}{7}}}$$

Easy to find that

$$f'(x)=\frac{(2x+\frac{1}{2}\sin 2x+x\cos 2x-2\sin 2x)x^6\left(\frac{\sin x}{x}\right)^{\frac{20}{7}}-(x^2+\frac{x}{2}\sin 2x-2\sin^2 x)\left[x^6\left(\frac{\sin x}{x}\right)^{\frac{20}{7}}\right]'}{x^{12}\left(\frac{\sin x}{x}\right)^{\frac{40}{7}}}$$

Where

$$\left[x^6\left(\frac{\sin x}{x}\right)^{\frac{20}{7}}\right]'=6x^5\left(\frac{\sin x}{x}\right)^{\frac{20}{7}}+x^6\frac{20}{7}\left(\frac{\sin x}{x}\right)^{\frac{13}{7}}\frac{x\cos x-\sin x}{x^2}$$

Thus it can be reasoned that $f'(x)=\dfrac{g(x)}{7x^{10}\left(\frac{\sin x}{x}\right)^{\frac{41}{7}}}$

where

$$g(x)=7x\sin^3 x(2x+x\cos 2x-\frac{3}{2}\sin 2x)-(x^2+\frac{x}{2}\sin 2x-2\sin^2 x)(22\sin^3 x+20x\sin^2 x\cos x)$$

$$=-21x^2\sin^3 x+6x^2\sin^5 x-3x\sin^4 x\cos x-20x^3\sin^2 x\cos x+44\sin^5 x$$

$$=\sin^5 x(6x^2+44-21x^2\csc^2 x-3x\cot x-20x^3\cot x\csc^2 x)$$

$$=\sin^5 x\cdot h(x)$$

where $h(x)=6x^2+44-21x^2\csc^2 x-3x\cot x-20x^3\cot x\csc^2 x$

By using(2.2)(2.4)(2.8),we can get

$$h(x)=6x^2+44-21x^2(\frac{1}{x^2}+\sum_{n=1}^{\infty}2^{2n}(2n-1)|B_{2n}|\frac{x^{2n-2}}{(2n)!})-3x(\frac{1}{x}-\sum_{n=1}^{\infty}2^{2n}|B_{2n}|\frac{x^{2n-1}}{(2n)!})$$

$$+20x^3(-\frac{1}{x^3}+\frac{1}{2}\sum_{n=2}^{\infty}2^{2n}(2n-1)(2n-2)|B_{2n}|\frac{x^{2n-3}}{(2n)!})$$

$$=6x^2+\sum_{n=1}^{\infty}(24-42n)2^{2n}|B_{2n}|\frac{x^{2n}}{(2n)!})+10\sum_{n=2}^{\infty}2^{2n}(2n-1)(2n-2)|B_{2n}|\frac{x^{2n}}{(2n)!})$$

$$=6x^2-6x^2+\sum_{n=2}^{\infty}(24-42n)2^{2n}|B_{2n}|\frac{x^{2n}}{(2n)!})+10\sum_{n=2}^{\infty}2^{2n}(2n-1)(2n-2)|B_{2n}|\frac{x^{2n}}{(2n)!})$$

$$=\sum_{n=2}^{\infty}a_n 2^{2n}|B_{2n}|\frac{x^{2n}}{(2n)!}=\sum_{n=3}^{\infty}a_n 2^{2n}|B_{2n}|\frac{x^{2n}}{(2n)!}$$

Where

$$a_n=10(2n-1)(2n-2)-42n+24=40n^2-102n+44$$

Let

$$a(x)=40x^2-102x+44$$
$$a'(x)=80x-102$$

Therefore, $a'(x)=80x-102>0$ holds true for any x such that $x>2$.

This implies that $a(x)$ is strictly increasing on $(2,+\infty)$.

Thus, when $x\in(2,+\infty)$, $a(x)>a(2)=0$

Which demonstrates that for all n>2,$a_n>0$.

Given the fact that all the coefficients of $h(x)$ are positive integers, $h(x)>0$ is true for every $x\in(0,\frac{\pi}{2})$, which would in turn prove that $g(x)>0$.

And as $7x^{10}(\frac{\sin x}{x})^{\frac{41}{7}}$ is surely greater than zero, this would indicate that $f'(x)$, while $x\in(0,\frac{\pi}{2})$, is also positive.

we can conclude that $f(x)$ is strictly increasing on $(0,\frac{\pi}{2})$

So, $f(x) > \lim_{x\to 0^+} f(x)$

Furthermore $\lim_{x\to 0^+} f(x) = \frac{2}{45}$,The proof of Theorem 4 is complete.

4 A Concise Proof of Theorem 1 and Theorem 2

4.1. A Concise Proof of Theorem1

Let $$f(x) = \frac{\sin^2 x}{x^5 \tan x} + \frac{1}{x^4} - \frac{2}{x^3 \tan x}$$

then $$f'(x) = \frac{g(x)}{x^6 \sin^2 x}$$

where $$g(x) = 2x^3 + 6x^2 \sin x \cos x - 5\sin^3 x \cos x - 3x\sin^2 x - 2x\sin^4 x$$

Direct calculation yields $g'(x) = 2\sin^3 x \cos x h(x)$

where $h(x) = 6x^2 \cot x \csc^2 x + 3x\csc^2 x - 9\cot x - 4x$

By using the power series expansion of (2.2)(2.4)(2.8),

$$h(x) = 6x^2[\frac{1}{x^3} - \frac{1}{2}\sum_{n=2}^{\infty}(2n-1)(2n-2)2^{2n}|B_{2n}|\frac{x^{2n-3}}{(2n)!}]$$

$$+3x[\frac{1}{x^2} + \sum_{n=1}^{\infty}(2n-1)2^{2n}|B_{2n}|\frac{x^{2n-2}}{(2n)!}]$$

$$-9[\frac{1}{x} - \sum_{n=1}^{\infty}2^{2n}|B_{2n}|\frac{x^{2n-1}}{(2n)!}] - 4x$$

$$= \sum_{n=2}^{\infty}[3-(2n-1)(2n-3)]3\cdot 2^{2n}|B_{2n}|\frac{x^{2n-1}}{(2n)!}$$

$$= \sum_{n=3}^{\infty}[3-(2n-1)(2n-3)]3\cdot 2^{2n}|B_{2n}|\frac{x^{2n-1}}{(2n)!} < 0$$

since $g'(x) = 2\sin^3 x \cos x h(x)$, when $x \in (0, \frac{\pi}{2}), \sin^3 x \cos x > 0$

So, $g'(x) = 2\sin^3 x \cos x h(x) < 0$

Then $g(x)$ is decreasing on $(0, \frac{\pi}{2})$. Now $g(0) = 0$, so, $g(x) < 0$

Therefore, $f'(x) = \dfrac{g(x)}{x^6 \sin^2 x} < 0$

So, $f(x)$ is strictly decreasing as x increases on $(0, \frac{\pi}{2})$.

At the same time,we find $\lim\limits_{x \to 0^+} f(x) = \dfrac{8}{45}$ and $\lim\limits_{x \to \frac{\pi}{2}^-} f(x) = \dfrac{16}{\pi^4}$. then the proof of theorem1 is complete.

4.2.A Concise Proof of Theorem2

Let $f(x) = \dfrac{2\sin x + \tan x - 3x}{x^4 \tan x}$

Thus the derivative of such a function could be expressed as follows:

$$f'(x) = \frac{(2\cos x + \sec^2 x - 3)x^4 \tan x - (2\sin x + \tan x - 3x)(4x^3 \tan x + x^4 \sec^2 x)}{x^8 \tan^2 x}$$

$$= \frac{1}{x^5 \tan^2 x} \cdot g(x)$$

where $g(x) = 2x\sin x - 8\sin x \tan x - 4\tan^2 x + 9x\tan x - 2x\sin x \sec^2 x + 3x^2 \sec^2 x$

$$= \frac{\sin^3 x}{\cos^2 x} h(x)$$

And where $h(x) = -2x - 8\cot x - 4\csc x + 9x\cot x \csc x + 3x^2 \dfrac{1}{\sin^3 x}$

By using the power series expansion of (2.2)(2.9)(2.10)(2.11), we can get

$$h(x) = -2x + 8\left(-\frac{1}{x} + \sum_{n=1}^{\infty} 2^{2n} |B_{2n}| \frac{x^{2n-1}}{(2n)!}\right) - 4\left(\frac{1}{x} + \sum_{n=1}^{\infty} (2^{2n} - 2)|B_{2n}| \frac{x^{2n-1}}{(2n)!}\right)$$

$$+9x\left(\frac{1}{x^2} - \sum_{n=1}^{\infty} (2^{2n} - 2)(2n-1)|B_{2n}| \frac{x^{2n-2}}{(2n)!}\right)$$

$$+3x^2\left(\frac{1}{x^3} + \frac{1}{2}\sum_{n=2}^{\infty} (2^{2n} - 2)(2n-1)(2n-2)|B_{2n}| \frac{x^{2n-3}}{(2n)!}\right)$$

$$+3x^2\left(\frac{1}{2x} + \frac{1}{2}\sum_{n=1}^{\infty} (2^{2n} - 2)|B_{2n}| \frac{x^{2n-1}}{(2n)!}\right)$$

$$= -\frac{1}{2}x + \sum_{n=1}^{\infty} [8 \cdot 2^{2n} - 4(2^{2n} - 2) - 9(2n-1)(2^{2n} - 2)]|B_{2n}| \frac{x^{2n-1}}{(2n)!}$$

$$+\frac{3}{2}\sum_{n=2}^{\infty} (2^{2n} - 2)(2n-1)(2n-2)|B_{2n}| \frac{x^{2n-1}}{(2n)!} + \frac{3}{2}\sum_{n=1}^{\infty} (2^{2n} - 2)|B_{2n}| \frac{x^{2n+1}}{(2n)!}$$

$$= -\frac{1}{2}x + \frac{1}{2}x + \sum_{n=2}^{\infty} [8 \cdot 2^{2n} - 4(2^{2n} - 2) - 9(2n-1)(2^{2n} - 2)]|B_{2n}| \frac{x^{2n-1}}{(2n)!}$$

$$+\frac{3}{2}\sum_{n=2}^{\infty} (2^{2n} - 2)(2n-1)(2n-2)|B_{2n}| \frac{x^{2n-1}}{(2n)!} + \frac{3}{2}\sum_{n=2}^{\infty} (2^{2n-2} - 2)(2n)(2n-1)|B_{2n-2}| \frac{x^{2n-1}}{(2n)!}$$

$$= \sum_{n=2}^{\infty} [8 \cdot 2^{2n} - 4(2^{2n} - 2) - 9(2n-1)(2^{2n} - 2) + \frac{3}{2}(2^{2n} - 2)(2n-1)(2n-2) + \frac{3}{2}(2^{2n-2} - 2)(2n)(2n-1)\frac{|B_{2n-2}|}{|B_{2n}|}]|B_{2n}| \frac{x^{2n-1}}{(2n)!}$$

$$= \sum_{n=2}^{\infty} d_n |B_{2n}| \frac{x^{2n-1}}{(2n)!}$$

Where,

$$d_n = 8 \cdot 2^{2n} - 4(2^{2n} - 2) - 9(2n-1)(2^{2n} - 2) + \frac{3}{2}(2^{2n} - 2)(2n-1)(2n-2) + \frac{3}{2}(2^{2n-2} - 2)(2n)(2n-1)\frac{|B_{2n-2}|}{|B_{2n}|}$$

$$= 4 \cdot 2^{2n} + 8 + 3(n-4)(2^{2n} - 2)(2n-1) + 3n(2^{2n-2} - 2)(2n-1)\frac{|B_{2n-2}|}{|B_{2n}|}$$

By careful calculation, $d_2 = 0, d_3 = 216 > 0$ can be gained.

When $n \geq 4, d_n > 0$ is true.

Then it can be reasonably obtained that

$$h(x) = \sum_{n=2}^{\infty} d_n |B_{2n}| \frac{x^{2n-1}}{(2n)!} = \sum_{n=3}^{\infty} d_n |B_{2n}| \frac{x^{2n-1}}{(2n)!} > 0$$

And thus $g(x) = \dfrac{\sin^3 x}{\cos^2 x} h(x) > 0$

As we know $f'(x) = \dfrac{1}{x^5 \tan^2 x} \cdot g(x) > 0$

So, $f(x)$ is strictly increasing on $\left(0, \dfrac{\pi}{2}\right)$,

At the same time, we find $\lim_{x \to 0^+} f(x) = \dfrac{3}{20}$ and $\lim_{x \to \frac{\pi}{2}^-} f(x) = \dfrac{16}{\pi^4}$.

Now, the proof of theorem2 is complete.

Acknowledgements

At the end of the paper, we would like to express our full appreciation to our families and friends who have supported us in a variety of ways. An enduring process has become less enduring with your support.

We would like to pay full gratitude to Mr. Zhu Ling, Mr. Chen Yizhi and Ms. Rui Yihe of Zhejiang Gongshang University for their help in the process of calculating and proving. It would be painfully difficult or perhaps impossible to finish this paper without you.

And we would also like to pay tribute to our respectable teachers---- Mr. Ni Ruixiang and Mr. Lu Dongbo of Hangzhou Foreign Languages School, and Mr. “Eric” Wang Yuheng of Nanson Education----for their informing us of such a competition and for their non-wavering belief in us.

Also, we wish to thank the committee of Yau Awards for providing such a valuable opportunity for us.

Last but not least, we would like to thank ourselves. We would thank our teammates for their efforts they have made, both in the researching of the sources and in upholding such a team. Together, we have journeyed beyond the boundaries of high school mathematics, we have explored something more advanced, we have overcome problems both individually and as a team. As an old saying goes: “When three is combined, strength is thus defined.” Let us pay the final respect to ourselves. Thank you, team.

References

[1]. Department of Mathematics of Tongji University, *Advanced Mathematics (Edition 7)*， Advanced Education Press, 2014.
[2]. J.B.Wilker, *E3306*, American Mathematical Monthly, 96(1989),no.1, 55.
[3]. J.S.Sumner, A.A.Jagers, M. Vowe, and J.Anglesio, *Inequalities Involving Trigonometric Function,* American Mathematical Monthly, 98(1991), No.3, 264-267.
[4]. L.Zhang and L.Zhu, *A New Elementary Proof of Wilker's Inequalities,* Mathematical Inequalities and Applications, Vol.11, No.1, pp. 149-151, 2008
[5]. L. Zhu, *Some New Wilker-Type Inequalities for Circular and Hyperbolic Functions*, Abstract and Applied Analysis, Vol.2009, Article ID 485842, 2009.
[6]. L. Zhu, *A New Simple Proof of Wilker's Inequality,* Mathematical Inequalities and Applications, Vol.8, No.4, pp. 749-750, 2005
[7]. M.Abramowitz, I.A.Stegun, (eds.), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Applied Mathematics Series, Vol.55, 9th printing, National Bureau of Standards, Washington(1972).
[8]. C.D'Aniello, *On some Inequalities for the Bernoulli Numbers*, Rend. Circ. Mat.Palermo 43, 329-332.
Retrieved from https://doi.org/10.1007/BF02844246
[9]. H.Alzer, Sharp Bounds for the Bernoulli Numbers, Arch.Math. 74, 207-211(2000).
Retrieved from https://doi.org/10.1155/2012/137507
[10]. F.Qi, A Double Inequality for the Ration of Two Non-Zero Neighbouring Bernoulli Numbers, J.Comput.Appl.Math.(2019, in press).
Retrieved from https://doi.org/10.1016/j.cam.2018.10.049
[11]. L.Zhu, Marija Nenezić, New Approximation Inequalities for Circular Functions, Journal of Inequalities and Applications, (2018)2018:313,
Retrieved from https://doi.org/10.1186/s13660-018-1910-9
[12]. T.Arakawa et al., *Bernoulli Numbers and Zeta Functions*, Springer Monographs in Mathematics, DOI10.1007/978-4-431-54919-2_1, 2014.
[13]. C. Huygens, *Oeuvres Completes 1888-1940,* Société Hollondaise des Science, Haga.
[14]. Chao-Ping Chen, Wing-Sum Cheung, *Sharpness of Wilker and Huygens Type Inequalities*, Journal of Inequalities and Applications 2012, 2012-72
[15]. L.Zhu, *Some New Inequalities of the Huygens Type*, Comput.Math.Appl.58 (2009), pp.1998-2004
[16]. Chao-Ping Chen, *Sharp Wilker- and Huygens-type Inequalities for Inverse Tirgonometric and Inverse Hyperbolic Functions*, Integral Transforms and Special Functions, Vol.23, No.12, December 2012, 865-873