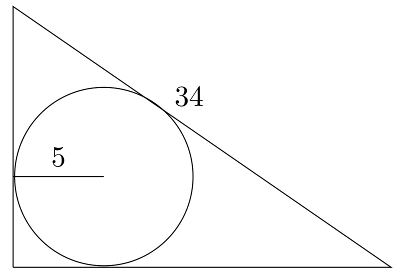
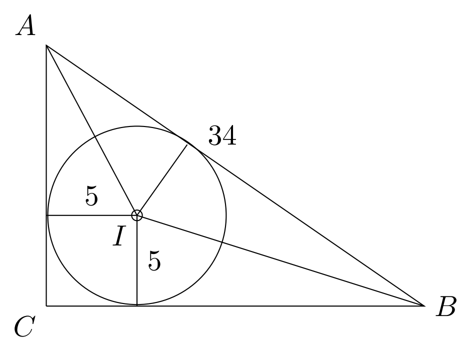
1. A circle with radius 5 is inscribed in a right triangle with hypotenuse 34 as shown below. What is the area of the triangle? Note that the diagram is not to scale.



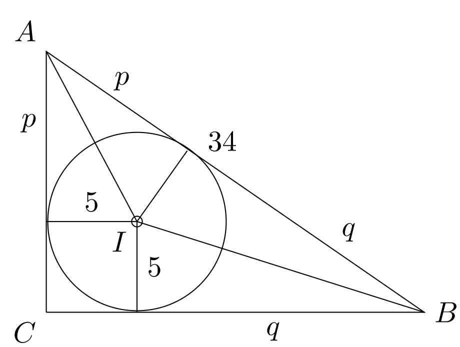
Answer: 195

Solution 1: We wish to use the formula for the area of a triangle: $A = rs$ where r is the radius of the in-circle and s is the semi-perimeter. To use this formula, all we need to do is compute the semi-perimeter.

The first thing to do is to draw all of the angle bisectors and radii of the triangle. We will also label the triangle ABC and the circle I for convenience.



We see that since the triangle is right, two radii and two of the sides make up a square. Let the distance from point A to the tangent point of circle I with line AB be p and the distance from point B the the tangent point of circle I with line BC equal to q. The perimeter is then $5 + 5 + p + q + 34$. Since the two tangents from any point to a circle are equal in length, we see that p is also the distance from A to the tangent point of circle I to line AB and q is the distance from B to the tangent point of circle I to line AB. This is shown below:



Therefore, $p+q$ must equal the length of the hypotenuse, or 34. Therefore the perimeter is $5+5+34+34=78$. The semi-perimeter is therefore 39 and the area of the circle is $rs = \boxed{195}$.

Solution 2: If a, b, c are lengths of a right triangle, then $\frac{a+b-c}{2}$ is the length of the radius of the incircle. Hence, letting $a = AC$ and $b = BC$, we wish to find $\frac{1}{2}ab$, the area of triangle ABC. We see that

$$\frac{a+b-34}{2} = 5 \implies a+b = 44$$

In addition, by the Pythagorean Theorem, we have

$$a^2 + b^2 = 34^2$$

By the formula $(a+b)^2 = a^2+b^2+2ab$, we obtain that $44^2 - 34^2 = 2ab \implies 780 = 2ab \implies \frac{1}{2}ab = \boxed{195}$ as desired.

2. For how many values of x does $20^x * 18^x = 2018^x$?

   Answer: 1

   Solution: The equation can be rearranged as $(\frac{360}{2018})^x = 1$, at which point the only possible value for x is 0. Thus, there is $\boxed{1}$ solution.

3. 2018 people (call them A, B, C, ...) stand in a line with each permutation equally likely. Given that A stands before B, what is the probability that C stands after B?

   Answer: $\frac{1}{3}$

   Solution: We know that the probability that A, B, and C stand in that order is $\frac{1}{6}$ since each permutation is equally likely. We also know that the probability that C stands after B is $\frac{1}{2}$ by symmetry. Therefore, the conditional probability is $\frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3}$.

4. Consider a standard (8-by-8) chessboard. Bishops are only allowed to attack pieces that are along the same diagonal as them (but cannot attack along a row or column). If a piece can attack another piece, we say that the pieces threaten each other. How many bishops can you place a chessboard without any of them threatening each other?

   Answer: 14

   Solution: It is helpful to draw a chessboard. Since I'm too lazy to find a scanner and importing pictures into LaTeX is annoying, I'll just describe what a chessboard looks like.

   A bishop which begins on a dark diagonal cannot move to a light diagonal and vice versa. Ergo, we shall only consider the dark diagonals (without loss of generality) and then multiply by 2.

   From Black's perspective (per the "queen starts on her color" rule) we have 8 diagonals facing to the far right (the first starts at a1 and the last starts at h8). We shall call these queenside diagonals, because the left, from the player's point of view, is their queenside. However there are 7 diagonals facing to the far left (the first starts at b8 and the last starts at h2). These shall be called kingside diagonals. Between any two light diagonals is a dark diagonal. This completely characterises the dark diagonals.

   If a bishop is placed onto a dark tile, it threatens all pieces on both dark diagonals passing through it. So place a bishop on a tile, and we are left with 7 queenside and 6 kingside diagonals. Repeating the process by placing a total of 7 bishops, and all kingside diagonals are occupied.

   So, we can put 7 bishops on the dark tiles, and 7 more on the light tiles. Therefore there are 14 bishops.

I want to play some chess now...

5. How many integers can be expressed in the form:

$$\pm 1 \pm 2 \pm 3 \pm 4 \cdots \pm 2018?$$

Answer: $(2019 * 1009) + 1 = 2037172$

Solution: Any odd number between $-1-2-\cdots-2018$ and $1+2+\cdots+2018 = 2019*1009$ work.

6. A rectangular prism with dimensions 20 cm by 1 cm by 7 cm with made with blue 1 cm unit cubes. The outside of the rectangular prism is coated in gold paint. If a cube is chosen at random and rolled, what is the probability that the size facing up is painted gold?

Answer: $\frac{167}{420}$

Solution: Every side has an equal probability of being face up. There are $20 \times 1 \times 7 = 140$ cubes, each with 6 sides; thus, there are 840 sides. There are $2 \times (20 \times 1 + 20 \times 7 + 7 \times 1) = 334$ gold sides. Thus, the probability of rolling a gold side is $\frac{334}{840} = \boxed{\frac{167}{420}}$

7. Suppose there are 2017 spies, each with $\frac{1}{2017}$th of a secret code. They communicate by telephone; when two of them talk, they share all information they know with each other. What is the minimum number of telephone calls that are needed for all 2017 people to know all parts of the code?

Answer: 4030

Solution: We use induction. We claim that for $n \geq 4$ spies the minimum number of telephone calls they need to know all parts of the code are $2n-4$. For $n=4$, person 1 calls person 2, person 3 calls person 4, person 1 calls person 3, and person 2 calls person 4. Now suppose for some $n > 4$ that fewer than $2n-4$ calls are required. Then without a loss of generality, let the call between 1 and n be the last phone call, let the call between 2 and n be the first phone call n has, and consider the phone calls made between all the people except for the nth person (note that n must have had at least 2 phone calls: one to communicate its information and another to receive its information). The phone call between 2 and n is unnecessary and the phone call between 1 and n is also unnecessary to communicate between the $n-1$ people. Hence, fewer than $2(n-1)-4$ phone calls were required for $n-1$ people. But this is a contradiction. Hence, $2n-4$ is optimal for n people. Thus, the answer is $2(2017)-4 = \boxed{4030}$.

8. Alice is playing a game with 2018 boxes, numbered $1-2018$, and a number of balls. At the beginning, boxes $1-2017$ have one ball each, and box 2018 has $2018n$ balls. Every turn, Alice chooses i and j with $i > j$, and moves exactly i balls from box i to box j. Alice wins if all balls end up in box 1. What is the minimum value of n so that Alice can win this game?

Answer: 2016

Solution: For $n < 2016$, it is impossible to get box 2017 to have an integer multiple of 2017 balls; thus, Alice is guaranteed to lose. For $n = 2016$, move all balls to box 2017. This yields 2017^2 balls in box 2017. Move $2*2017$ balls to box 1, then repeat this process through induction.

9. Circles A, B, and C are externally tangent circles. Line PQ is drawn such that PQ is tangent to A at P, tangent to B at Q, and does not intersect with C. Circle D is drawn such that it passes through the centers of A, B, and C. Let R be the point on D furthest from PQ. If A, B,and C have radius $3, 2$,and 1, respectively, the area of triangle PQR can be expressed in the form of $a+b\sqrt{c}$, where a, b,and c are integers with c not divisible by any prime square. What is $a+b+c$?

   Answer: 11

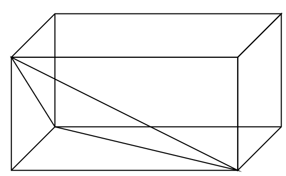
   Solution: Note that the triangle formed by the centers of A, B, C form a $3-4-5$ triangle. Thus, D is a circle of radius 2.5, with the centers of A and B forming a diameter of D. Since the center of A is 3 from PQ, and the center of B is 2 from PQ, the midpoint of AB, which is the center of D, is 2.5 from PQ. Thus, D is tangent to PQ, and R is 5 from PQ. Since segment PQ has length $\sqrt{24}$, triangle PQR has area $0+5\sqrt{6}$. This gives $a+b+c=11$.

10. A rectangular prism has three distinct faces of area 24, 30, and 32. The diagonals of each distinct face of the prism form sides of a triangle. What is the triangle's area?

    Answer: 25

    Solution:

    We note that we can represent the triangle in the context of the rectangular prism with lengths a, b and c as follows:



    We see that the vertices of the triangle lie on points $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$. To calculate the area of this triangle, we can take half the magnitude of the cross-product between two of the vectors that make up the triangle. These vectors are $(a, -b, 0)$ and $(a, 0, -c)$. The cross product is $(bc)\vec{i} + (ac)\vec{j} + (ab)\vec{k}$ which has magnitude $\sqrt{(ab)^2 + (bc)^2 + (ac)^2}$. These values are exactly the areas of the faces of the rectangle. So we have $\sqrt{24^2 + 30^2 + 32^2} = 50$. Since the area of the triangle is the cross product divided by 2, we have out answer of 25.

11. Ankit, Box, and Clark are playing a game. First, Clark comes up with a prime number less than 100. Then he writes each digit of the prime number on a piece of paper (writing 0 for the tens digit if he chose a single-digit prime), and gives one each to Ankit and Box, without telling them which digit is the tens digit, and which digit is the ones digit. The following exchange occurs:

    - Clark: There is only one prime number that can be made using those two digits.
    - Ankit: I don't know whether I'm the tens digit or the ones digit.
    - Box: I don't know whether I'm the tens digit or the ones digit.
    - Box: You don't know whether you're the tens digit or the ones digit.
    - Ankit: I don't know whether you're the tens digit or the ones digit.

    What was Clark's number?

    Answer: 11

Solution: The first statement after Clark rules out the possibility of Ankit having $0, 4, 6, 8$. The second rules out the possibility of Box having $0, 4, 6, 8, 2, 5$, as Ankit doesn't have 0. The third means that Box knows Ankit doesn't have 2 and 5, as that would immediately mean that 2 or 5 was the tens digit. This means he doesn't have 3 or 9; the only primes with tens digit as 2 or 5 remaining are $23, 29, 53, 59$. Additionally, Ankit doesn't have 2 or 5. At this point, the only primes remaining are $11, 13, 17, 19, 31, 37, 71, 73, 79, 97$. However, if 13 was Clark's number, it would be impossible for Ankit and Box to determine if his number was 13 or 31. Similarly, they can remove most cases, bringing it down to two possibilities: 11 and 19. Thus, Box has 1. If Clark's number was 19, it would be possible for Ankit to know Box's digit is the tens digit. Thus, the fourth statement rules out 19, and Clark's number is 11.

12. Let $f : [0, 1] \to \mathbb{R}$ be a monotonically increasing function such that

$$f\left(\frac{x}{3}\right) = \frac{f(x)}{2}$$

$$f(1 - x) = 2018 - f(x)$$

If $f(1) = 2018$, find $f\left(\frac{12}{13}\right)$.

Answer: $\mathbf{\frac{12108}{7}}$

Solution: Let us scale this down from 2018 to 1, so that $f(1) = 1$, and that $f(1-x) = 1-f(x)$. Notice that $12708 = 2018 \times 6$. So in this case, $f\left(\frac{12}{13}\right) = \frac{6}{7}$.

Lemma. *Suppose there exists a base 3 representation of x that is*

$$\sum_{i=1}^{\infty} \frac{a_i}{3^i}$$

where $a_i \neq 1$. Then

$$f(x) = \sum_{i=1}^{\infty} \frac{a'_i}{2^i}$$

where $a'_i = 1$ if $a_i = 2$, and 0 otherwise.

Proof. We shall show this for rational x for which the denominator is a power of 3. Note that $f(\frac{1}{3}) = \frac{1}{2}$, since

$$\frac{1}{3} = \sum_{i=2}^{\infty} \frac{2}{3^i}$$

$$f(\frac{2}{3}) = \frac{1}{2}$$

$$f(0) = 0$$

Now suppose this is true for x with N decimal places are not all zero (and the rest are all 0). We claim that this is true for $N+1$. Then $x = \frac{2}{3^{N+1}} + y$ for some y with denominator 3^N. Note that we can factor $x = \frac{1}{3}\left(\frac{2}{3^N} + 3y\right)$. We remark that $f(3x) = 2f(x)$ where the $3x$ is taken modulo 1. Hence, $f(\frac{x}{3}) = \frac{1}{2}\sum \frac{a_i}{2^i}$, where $a_{N+1} = 1$. Hence, by induction, we're done with the finite case. To show the infinite, approximate it by finite cases and use the fact that f is monotonic. $\square$

Note that $\frac{12}{13} = 0.220220220220... = 0.\overline{220}_3$ in base 3. Hence, under the image of f, it is

$$0.110110110... = \frac{6}{7}$$

Now scaling up by 2018, we get $\boxed{\frac{12108}{7}}$ as desired.

13. Find the value of

$$\frac{1}{\sqrt{2}^1} + \frac{4}{\sqrt{2}^2} + \frac{9}{\sqrt{2}^3} \cdots$$

Answer: $\mathbf{24 + 17\sqrt{2}}$

Solution: Let's start by replacing $\frac{1}{\sqrt{2}}$ with x for simplicity; we can replace it at the end. We then call the sum S. This summation then becomes $S = \sum_{n=1}^{\infty}(n^2)(x^n) = 1^2x^1 + 2^2x^2 + 3^2x^3 + 4^2x^4...$ We then notice that the common difference between square numbers is the odd numbers (e.g. $2^2 - 1^2 = 3, 3^2 - 2^2 = 5, 4^2 - 3^2 = 7$, etc.), which we want to obtain, so with a bit of algebraic manipulation we find that

$$\begin{aligned} S &= 1^2x^1 + 2^2x^2 + 3^2x^3 + 4^2x^4... \\ -xS &= \qquad - 1^2x^2 - 2^2x^3 - 3^2x^4... \\ (1-x)S = S - xS &= 1x^1 + 3x^2 + 5x^3 + 7x^4... \end{aligned}$$

Now that the terms are increasing by a factor of two, we want to do manipulate it one more time to make the coefficients constant: this makes the equation even easier to work with. We notice that the common difference between the terms is now 2, and we do the same process again:

$$\begin{aligned} (1-x)S &= 1x^1 + 3x^2 + 5x^3 + 7x^4... \\ -x(1-x)S &= \qquad - 1x^2 - 3x^3 - 5x^4... \\ (1-x)(1-x)S &= 1x^1 + 2x^2 + 2x^3 + 2x^4... \\ (1-x)^2S &= x + 2(x^2 + x^3 + x^4...) \end{aligned}$$

The only part of the equation left to simplify before we can plug x back in to solve for S is the infinite series $x^2 + x^3 + x^4...$. Once again, we want to simplify the coefficients; but now that their common difference is zero, the simplification nullifies the series altogther, which is extremely helpful for computation. Let us call $s = x^2 + x^3 + x^4...$, and we can simplify it like this:

$$\begin{aligned} s &= x^2 + x^3 + x^4... \\ -xs &= \qquad - x^3 - x^4... \\ x - xs = (1-x)s &= x^2 \\ s &= \frac{x^2}{1-x} \end{aligned}$$

We are now ready to substitute $\frac{1}{\sqrt{2}}$ back in for x, and solve for S.

$$\begin{aligned}
(1-x)^2 S &= x + 2(x^2 + x^3 + x^4 \ldots) \\
(1-x)^2 S &= x + 2s \\
\left(1 - \frac{1}{\sqrt{2}}\right)^2 S &= \frac{1}{\sqrt{2}} + 2\frac{\left(\frac{1}{\sqrt{2}}\right)^2}{1 - \frac{1}{\sqrt{2}}} \\
\left(\frac{\sqrt{2}-1}{\sqrt{2}}\right)^2 S &= \frac{\sqrt{2}}{2} + 2\frac{\frac{1}{2}}{\frac{\sqrt{2}-1}{\sqrt{2}}} \\
\frac{2 - 2\sqrt{2} + 1}{2} S &= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{\sqrt{2}-1} \\
\frac{3 - 2\sqrt{2}}{2} S &= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}(\sqrt{2}+1)}{(\sqrt{2}-1)(\sqrt{2}+1)} \\
\frac{3 - 2\sqrt{2}}{2} S &= \frac{\sqrt{2}}{2} + \frac{2+\sqrt{2}}{1} \\
(3 - 2\sqrt{2}) S &= \sqrt{2} + 2(2 + \sqrt{2}) \\
S &= \frac{4 + 3\sqrt{2}}{3 - 2\sqrt{2}} = \frac{(4+3\sqrt{2})(3+2\sqrt{2})}{(3-2\sqrt{2})(3+2\sqrt{2})} \\
S &= \frac{12 + 17\sqrt{2} + 12}{9 - 8} \\
S &= \boxed{24 + 17\sqrt{2}}
\end{aligned}$$

14. Let $F_1 = 0, F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$. Compute

$$\sum_{n=1}^{\infty} \frac{\sum_{i=1}^{n} F_i}{3^n}$$

Answer: $\mathbf{\frac{9}{10}}$

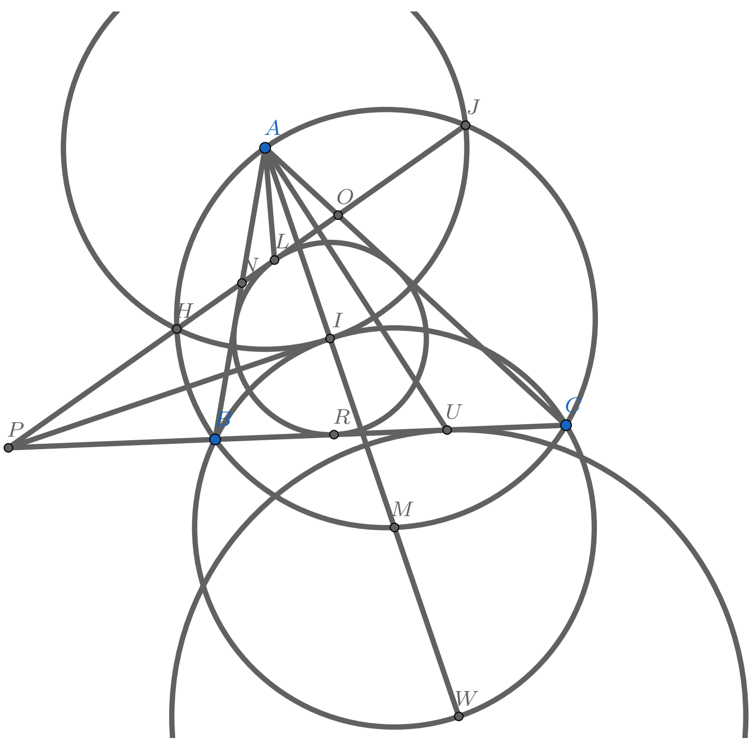
Solution: More fleshed-out:

$$\begin{aligned}
\sum_{n=0}^{\infty}\sum_{i=1}^{n}\frac{F_i}{3^n} &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{F_i}{3^{i+j}} \\
&= \sum_{i=0}^{\infty}\frac{F_i}{3^i}\sum_{j=0}^{\infty}\frac{1}{3^j} \\
&= \left(\sum_{i=0}^{\infty}\frac{F_i}{3^i}\right)\left(\sum_{j=0}^{\infty}\frac{1}{3^j}\right) \\
&= \left(\frac{1}{1}\cdot\frac{\frac{1}{3}}{1-\left(\frac{1}{3}^2+\frac{1}{3}\right)}\right)\left(\frac{1}{1-\frac{1}{3}}\right) \\
&= \frac{1}{1}\cdot\frac{3}{5}\cdot\frac{3}{2} \\
&= \boxed{\frac{9}{10}}.
\end{aligned}$$

15. Let triangle ABC have side lengths $AB = 13$, $BC = 14$, $AC = 15$. Let I be the incenter of ABC. The circle centered at A of radius AI intersects the circumcircle of ABC at H and J. Let L be a point that lies on both the incircle of ABC and line HJ. If the minimal possible value of AL is $\sqrt{n}$, where $n \in \mathbb{Z}$, find n.

Answer: 17

Solution:



The first thing to observe here is that HJ is in fact tangent to the incircle of ABC at L. To prove this, we see that A is the circumcenter of triangle HIJ, so by the incenter-excenter lemma, I is the incenter of triangle HJM. In addition, observe that by Euler's theorem that $d^2 = R^2 - 2rR$, so the inradius of HJM equal to the inradius of ABC.

Let HJ intersect AB at N and AC at O. The main observation is that triangle AON is similar to triangle ABC. To prove this, observe that by power of a point theorem, $AN \times NB = HN \times NJ = AI^2 - AN^2 \implies AN(NB + AN) = AN \times AB = AI^2$. Similarly, $AO \times AC = AI^2$. Hence, $AN \times AB = AO \times AC \implies \frac{AN}{AC} = \frac{AO}{AB}$. Thus, by SAS similarity, triangle AON is similar to triangle ABC. Let the A-excenter of triangle ABC be tangent to BC at U and the incircle of ABC to be tangent at BC at R. It is well known that $BR = UC = s - b = 6$. Since the height from A of the triangle is 12, we have by the pythagorean theorem that

$$12^2 + (14 - 6 - 5)^2 = AU^2 \implies AU = \sqrt{153} = 3\sqrt{17}$$

In addition, it is well known that the ratio of the in-radius to the A-exradius is $\frac{s-a}{s} = \frac{7}{21} = \frac{1}{3}$. Hence, $AL = \frac{AU}{3} = \sqrt{17}$, so the answer is $\boxed{17}$.