TEAM ROUND SOLUTIONS

1. Notice for each real number x, $1-x$ also has a repeating cycle. Thus, we can pair up all our numbers, so it suffices to count the number of arrangments of distinct digits, and divide by two. First, we choose a nonzero subset of the digits $\{0,1,2,3,4,5\}$. Then, if k is the size of our subset, we have $k!$ ways of arranging those digits, thus, the number of ways is $\sum_{i=1}^{6}\binom{6}{i}i!$, which is equal to $6+15\cdot 2+20\cdot 6+15\cdot 24+6\cdot 120+720=1956$. We divide by two to get the sum equal to $\boxed{978}$

2.
$$\begin{aligned}\prod_{k=1}^{254}\log_{k+1}(k+2)^{u_k} &= \prod_{k=1}^{254}u_k\prod_{k=1}^{254}\log_{k+1}(k+2)\\ &= \prod_{k=1}^{254}u_k\prod_{k=1}^{254}\frac{\log(k+2)}{\log(k+1)}\\ &= \prod_{k=1}^{254}u_k\cdot\frac{\log 2}{\log 256}\\ &= 8\prod_{k=1}^{254}u_k\\ &= 8\left((-1)\cdot\frac{1}{1}\cdot(-3)\cdot\frac{1}{3}\cdot\ldots\cdot(-253)\cdot\frac{1}{253}\right)\\ &= 8\cdot(-1)^{127}=-8.\end{aligned}$$

3. Label the sides of the triangle a,b,c, and the angles α,β,γ. We have the squared lengths S_1,S_2,S_3 to be
$$S_1^2=a^2 \qquad S_2^2=b^2 \qquad S_3^2=c^2$$
The squared lengths of S_4,S_5,S_6 are
$$S_4^2=c^2+b^2+2cb\cos(\alpha) \qquad S_5^2=a^2+c^2+2ac\cos(\beta) \qquad S_6^2=a^2+b^2+2ab\cos(\gamma)$$
Thus,
$$\frac{S_4^2+S_5^2+S_6^2}{S_1^2+S_2^2+S_3^2}=\frac{2(a^2+b^2+c^2)+2ab\cos(\gamma)+2bc\cos(\alpha)+2ac\cos(\beta)}{a^2+b^2+c^2}$$
But the above expression can be simplified by using law of cosines even more, by substituting $2ab\cos(\gamma)=a^2+b^2-c^2$, and equivalently for the other two terms, and we get it to simplify to
$$\frac{2(a^2+b^2+c^2)+a^2+b^2+c^2}{a^2+b^2+c^2}=\boxed{3}$$

4. Let p be the probability that a mission with the spy fails, and let n be the total number of people. Let A_i be the event person i is the spy, and let B the event that the first $n-1$ missions succeed, but the last mission fails. Then, we want to compute $Pr[A_n|B]=\frac{Pr[B|A_n]Pr[A_n]}{Pr[B]}$. The probability of B can be computed by considering cases. The probability of B is given by $\sum_{i=1}^{n}Pr[B|A_i]Pr[A_i]$. But $Pr[A_i]=\frac{1}{n}$ symmetrically, and $Pr[B|A_i]=(1-p)^{n-i}p$, since the first $i-1$ missions are guaranteed to succeed, the next $N-i$ missions need to succeed, and the last mission fails. Thus. $Pr[B]=\frac{p}{n}\sum_{i=1}^{n}(1-p)^{n-i}=\frac{p}{n}\frac{1-(1-p)^n}{p}$.
Thus, $Pr[A_n|B]=\frac{p\frac{1}{n}}{\frac{1-(1-p)^n}{n}}=\frac{p}{1-(1-p)^n}$
When $p=1/2, n=12$, we have $0.5/(1-(1/2)^{12})=2^{11}/2^{12}-1=2048/4095$

5. Let $n=10a+d$, so $f(n)=a+md$. We want $10a+d\equiv 0 \pmod p \implies a+md\equiv 0 \pmod p$, so, we have $d\equiv -10a \pmod p$. Plugging this in to our second equation, we have $a(1-10m)\equiv 0 \pmod p$, which we want to be true for any a, so we must have $10m\equiv 1 \pmod p$. Thus we want the inverse of 10 modulo 2013 which is our value of m, which can be found by $\frac{2013\cdot 3+1}{10}=\boxed{604}$.

6. Consider coordinates. Circle a will have diameter p, and circle b will have diameter $1-p$. The diameter of circle s is given by the height of the circle at point p, which is just $\sqrt{1-p^2}$. Circle t will have the same diameter. Thus, we must have $A(s)+A(t)=2\left(\frac{1-p^2}{2}\right)^2\pi$, and $A(a)+A(b)=\left(\frac{p}{2}\right)^2\pi+\left(\frac{1-p}{2}\right)^2$, so the ratio is equal to $\frac{2(1-p^2)}{2p^2-2p+1}$. When $p=1/42$, we have $1763/841$.

7. Without loss of generality, assume Bob travels with speed 1. Call Alice's same-line-segment speed a_2 and her different-line-segment speed a_1.
Suppose that Alice is at the start of a line segment of length y, and Bob is a distance $x < y$ along the same line segment. Then, when Bob has reached the end of this segment, Alice will have traveled $(y-x)*a_2$, and Bob and Alice will now be $y(1-a_2)-a_2x$ apart.
Then, Alice will spend $\frac{y(1-a_2)-a_2x}{a_1}$ time to travel to the beginning of the next segment. During this time, Bob will have traveled $\frac{y(1-a_2)-a_2x}{a_1}$ and thus they will be $\frac{y(1-a_2)-a_2x}{a_1}$ apart.
Now, let x_i be the distance apart Alice and Bob are when Alice is at the beginning of the segment with length $\frac{1}{2^i}$ (and we note that if $x_i > \frac{1}{2^i}$, then Alice can never catch up). Then we have $x_0 = 1$,

$$x_i = \frac{\frac{1}{2^{i-1}}(1-a_2)+a_2x_{i-1}}{a_1}$$

assuming that $x_i \leq \frac{1}{2^i}$ at all times. We claim that this holds true when $x_0 \leq \frac{2(a_2-1)}{2a_2-a_1}$. In fact, we claim that if $x_0 \leq \frac{2(a_2-1)}{2a_2-a_1}$, then $x_i \leq \frac{2(a_2-1)}{2^i(2a_2-a_1)}$ and prove this by induction. Suppose it holds for all $1, ..., i-1$, and that $x_0, ..., x_{i-1} > 0$. Then we have

$$\begin{aligned} x_i &= \frac{\frac{1}{2^{i-1}(1-a_2)}+a_2x_{i-1}}{a_1} \\ &\leq \frac{1-a_2}{2^{i-1}a_1} + \frac{a_2}{a_1}\cdot\frac{2(a_2-1)}{2^{i-1}(2a_2-a_1)} \\ &= \frac{(1-a_2)(2a_2-a_1)+2a_2(a_2-1)}{a_12^{i-1}(2a_2-a_1)} \\ &= \frac{(a_2-1)}{2^{i-1}(2a_2-a_1)}. \end{aligned}$$

Thus Alice gets arbitrarily close to Bob as they move along the path; at some point at or before S, then, Alice will catch up to Bob. when $a_2 = 1.28, a_1 = 0.6$, then we have $28/75$.

8. A well known formula for the totient function $\varphi(n)$ is $n\prod_{p|n}\left(1-\frac{1}{p}\right)$, where p is a prime. Using this, we can compute $\phi(13^r)$ for any natural number r.
We can split this into two parts, one part where we still have a factor of 13, and when we don't. We'll calculate the number of steps in each case.
   - Phase 1: There is still a factor of 13. 13 will decompose to $2^2\cdot 3$ under the totient function. Also, 3 will go to 2, and 2 will go to 1, so we can conclude after $k \leq r$ steps, the number is in the form $2^{2k}\cdot 3\cdot 13^{r-k}$. After r steps, we have the number $2^{2r}\cdot 3$.
   - Phase 2: First, one step will convert the number to 2^{2r}. After this, it will take $2r$ steps to remove all the factors of two one by one. This phase takes $2r+1$ steps.

   Thus, the total number of steps is $3r+1$, and when $r = 2012$, we have 6037 steps.

9. The first step is to come up with an algorithm. Clearly, we can always do this in n moves, just take the largest element, move it to the front, then the next largest, and so on until the list is sorted. However, we can do better. We don't actually have to move the largest element to the front. Instead, what we could do is take the second largest element, then the third largest, and so on, and the list wil be sorted. This algorithm takes $n-1$ moves for any list.
However, if the second largest and largest elements are already ordered (i.e. the second largest appears before the largest), we don't actually have to move them. That means it really only takes $n-2$ moves. We can extend this argument. Let us say the top k elements are already partially ordered. Then, the algorithm will take $n-k$ moves. Thus, it suffices to find the expected number of elements that are already partially ordered.
To show we can't do better, we know the number of elements that are out of place is $n-k$, and since we need to process them at least once, $n-k$ is our lower bound. Thus, the optimal algorithm takes $n-k$ steps.
Now, to find the expected number of partially ordered elements, we can see 1 element that is already in order. The probablity that $n-1$ and n are ordered in increasing order is $\frac{1}{2!}$, and in general, the probability that the last k numbers are ordered in increasing order is $\frac{1}{k!}$. Thus, we have the expected length of the

increasing sequence equal to $1 \cdot 1 + 2 \cdot \frac{1}{2!} + 3 \cdot \frac{1}{3!} + 4 \cdot \frac{1}{4!} + 5 \cdot \frac{1}{5!} + 6 \cdot \frac{1}{6!} = \frac{163}{60}$. The expected number of swaps is just $6 - x$, where x is the expected length of the increasing sequence, so our answer is $\boxed{\frac{197}{60}}$

10. Let M be the number of coins, k be the number of coins flipped per iteration, and N be the number of iterations performed.
Let us focus on one coin only. We want to find the probability that it is heads after N iterations. This is just the probability that the coin was flipped an even number of times. Let p be the probability that the coin is flipped in one iteration. Then, we just count the number of ways the coin could be flipped 0 times, 2 times, and so on. The expression is
$$\binom{N}{0}p^0(1-p)^N + \binom{N}{2}p^2(1-p)^{N-2} + \binom{N}{4}p^4(1-p)^{N-4} + \ldots + \binom{N}{\lfloor \frac{N}{2} \rfloor}p^{\lfloor \frac{N}{2} \rfloor}(1-p)^{N-\lfloor \frac{N}{2} \rfloor}$$
It is possible to reduce this expression. The coefficient of p^m can be found by the sum
$$\sum_{k=0}^{\lfloor m/2 \rfloor} \binom{N}{2k}(-1)^{m-2k}\binom{N-2k}{m-2k}$$
Notice that the $(-1)^{m-2k} = (-1)^m$ which is constant regardless of k, so we pull it out. Also, we can get a tidier expression. We have $\binom{N}{2k}\binom{N-2k}{m-2k} = \frac{(N)!}{(N-2k)!(2k)!}\frac{(N-2k)!}{(N-m)!(m-2k)!}\frac{m!}{m!} = \binom{N}{m}\binom{m}{2k}$. Since the first term doesn't depend on k, we can pull it out to get the sum
$$(-1)^m\binom{N}{m}\sum_{k=0}^{\lfloor m/2 \rfloor}\binom{m}{2k}$$
But, this is easy to calculate, since it's the number of even sized subsets of m elements, which is just simply 2^{m-1} (note for $m = 0$, we have a special case that this is 1). Thus,
$$\sum_{k=0}^{\lfloor m/2 \rfloor}\binom{N}{k}(-1)^{m-2k}\binom{N-2k}{m-2k} = (-1)^m 2^{m-1}\binom{N}{m}$$
which is the coefficient of p^m in the sum. However, note the argument only works for $m > 0$, so we'll need other means to compute the coefficient for p^0. However, this is easy. The only contribution comes from the first term, which is simply 1. Thus, we have our sum is equivalent to
$$1 - 1\binom{N}{1}p^1 + 2\binom{N}{2}p^2 - 4\binom{N}{3}p^3 + \ldots - (-2)^{N-1}\binom{N}{N}p^N$$
We can rearrange this sum a bit in order to compress it. Rearranging gives us
$$\frac{1}{2}\left(1 + 1 - \binom{N}{1}2^1p^1 + \binom{N}{2}2^2p^2 - \binom{N}{3}2^3p^3 + \ldots + \binom{N}{N}(-2)^Np^N\right)$$
This reduces to
$$\frac{1}{2}\left(1 + (1-2p)^N\right)$$
Using linearity of expectation, the expected number of heads is thus
$$\frac{M}{2}\left(1 + (1-2p)^N\right)$$
We note that $p = \frac{k}{M}$, thus, we have
$$\frac{M}{2}\left(1 + \left(1 - \frac{2k}{M}\right)^N\right)$$
$$= \frac{1 + (M-2k)^N}{2M^{N-1}}$$
When $M = 728$, $N = 4001$ and $k = 314$, we obtain
$$= \frac{1 + 100^{4001}}{2 \cdot 728^{4000}}.$$
The top and bottom are relatively prime because $728 = 8 \cdot 7 \cdot 13$, and the top is obviously odd. It is also not divisible by 7 or 13. Thus we have
$$1 + 100^{4001} + 2 \cdot 728^{4000} \mod 10000 \equiv 1 + 2 \cdot 728^{4000}.$$

Since $728^{4000} \equiv 0 \mod 16$, it now suffices for us to find $728^{4000} \mod 625$. This is equal to 1 because $\varphi(625) = 500 | 4000$ and $\gcd(728, 625) = 1$. Then by the Chinese Remainder Theorem, $728^{4000} \equiv 9376 \mod 10000$ and our answer is

$$2 * 9376 + 1 = 8753.$$