1. Alice is planning a trip from the Bay Area to one of 5 possible destinations (each of which is serviced by only 1 airport) and wants to book two flights, one to her destination and one returning. There are 3 airports within the Bay Area from which she may leave and to which she may return. In how many ways may she plan her flight itinerary?

   Solution. *Alice has 3 ways of choosing a departing airport, 5 ways of choosing a location from there, and 3 ways to choose a returning airport, giving us* $3 \cdot 5 \cdot 3 = 45$ *possible flight itineraries.*

2. Determine the largest integer n such that 2^n divides the decimal representation given by some permutation of the digits 2, 0, 1, and 5. (For example, 2^1 divides 2150. It may start with 0.)

   Solution. *The largest integer we can generate with such permutations is* 5210*, so clearly* $n <= 11$*. We see that* $2048 \cdot 2 = 4096$ *and thus* $n = 11$ *is not possible. However,* 2^{10} *divides* 5120*, so we have* $n = 10$*.*

3. How many rational solutions are there to $5x^2 + 2y^2 = 1$?

   Solution. *Note that this is equal to the number of minimal solutions to* $5x^2+2y^2 = z^2$*, where* x, y, z *are integers. Taking this equation mod 5, we get that* $2y^2 = z^2$*. But 2 is a quadratic nonresidue modulo 5, so there are no solutions.*

4. Determine the greatest integer N such that N is a divisor of $n^{13} - n$ for all integers n.

   Solution. *Using Fermat's Little Theorem, we see that* $n^{13} \equiv n$ *mod 2,3,5,7, and 13. Thus* $2730 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 13$ *divides* N*. Meanwhile,* $2^{13} - 2 = 3 \cdot 2730$*. Since* $3^{13} - 3$ *is not divisible by 9, we must have* $N = 2730$*.*

5. Three balloon vendors each offer two types of balloons − one offers red & blue, one offers blue & yellow, and one offers yellow & red. I like each vendor the same, so I must buy 7 balloons from each. How many different possible triples (x, y, z) are there such that I could buy x blue, y yellow, and z red balloons?

   Solution. *First, we note that* (x, y, z) *is an ordered partition of 21. By stars and bars, there are* $\binom{23}{2} = 253$ *of those. However, since each color is not sold by one of the vendors, we must have* $x, y, z \le 14$*. Now, there are* $\binom{8}{2} = 28$ *partitions where* $x \ge 15$*. Since no two of* x, y, z *can simultaneously be* ≥ 15*, we have* $253 - 3 \cdot 28 = 169$ *partitions where no part is greater than 14.*

   Now, we claim that all of these remaining partitions are indeed possible. Suppose we have (x, y, z) *satisfying the conditions. Without loss of generality, assume that* $x \le 7$ *(by pigeonhole) and that* y *is the largest of the three. Then, buy 7 yellow balloons from the blue and yellow vendor, and* x *blue balloons from the blue and red vendor. Then, buy* $7-x$ *red balloons from the blue and red vendor - this is possible because otherwise* $z + x \le 7$*, and so* $y > 14$ *which contradicts our assumption. Finally, we buy the remainder of the balloons from the yellow and red vendor. This gives us a valid way to buy balloons satisfying* (x, y, z)*, and thus we have 169 feasible possibilities of balloon buying.*

6. There are 30 cities in the empire of Euleria. Every week, Martingale City runs a very well-known lottery. 900 visitors decide to take a trip around the empire, visiting a different city

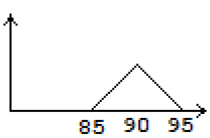
each week in some random order. 3 of these cities are inhabited by mathematicians, who will talk to all visitors about the laws of statistics. A visitor with this knowledge has probability 0 of buying a lottery ticket, else they have probability 0.5 of buying one. What is the expected number of visitors who will play the Martingale Lottery?

Solution. *Consider a random visitor Eve. Eve's trip is a permutation of the 30 cities, and since she visits them in a random order, each possible permutation is equally likely. Restricting out focus to the ordering of Martingale City and the three mathematician cities, we see that Eve has a* $\frac{1}{4}$ *chance of visiting Martingale City before the other three. In this case, she has a* $\frac{1}{2}$ *chance of buying a lottery ticket; in the other cases, she has probability 0 of buying one. Thus, Eve has a* $\frac{1}{8}$ *chance in total of buying a lottery ticket.*

This holds independently for all 900 visitors, giving us an expected value of $\frac{225}{2}$ *lottery tickets sold.*

7. At Durant University, an A grade corresponds to raw scores between 90 and 100, and a B grade corresponds to raw scores between 80 and 90. Travis has 3 equally weighted exams in his math class. Given that Travis earned an A on his first exam and a B on his second (but doesn't know his raw score for either), what is the minimum score he needs to have a 90% chance of getting an A in the class? Note that scores on exams do not necessarily have to be integers.

Solution. *The probability distribution of Travis's average score after the first two exams is as follows:*



where the area underneath the graph must be 1. Thus, the height of the triangle is $\frac{1}{5}$. *To find the 10th percentile of this graph, we look for a value x for which the area it encloses is* $\frac{1}{10}$, *i.e.* $x \cdot \frac{x}{25} \cdot \frac{1}{2} = \frac{1}{10}$. *This gives us* $x = \sqrt{5}$. *Travis will then get an A in the class if* $\frac{2(85+\sqrt{5})+n}{3} \geq 90$, *or* $170 + 2\sqrt{5} + x \geq 270$. *This gives* $x \geq 100 - 2\sqrt{5}$.

8. Two players play a game with a pile with N coins is on a table. On a player's turn, if there are n coins, the player can take at most $n/2+1$ coins, and must take at least one coin. The player who grabs the last coin wins. For how many values of N between 1 and 100 (inclusive) does the first player have a winning strategy?

Solution. *We claim that the second player can only win when* $N = 3(2^k - 1), k \geq 1$. *To show this, we use induction. First, we see easily that player 1 wins when* $N = 1$ *and* $N = 2$ *by grabbing all the coins, but they cannot win when* $N = 3$. *Now, suppose we have found the winning strategy for all* $k < N$. *Let* $n = 3(2^m - 1)$ *be the largest such number that is less than* N. *Then if* $N - (N/2+1) <= n$, *the first player can take away enough coins so that there are n left on the table, and then they will win. But the statement* $N - (N/2+1) <= n$ *is equivalent to* $N/2 - 1 \leq n$, *which means* $N \leq 2(3(2^m-1)+1) = 2(3 \cdot 2^m - 2) = 3(2^{m+1}-1)-1$. *Thus the next value of N for which the second player can win is* $N = 3(2^{m+1}-1)$, *completing the induction. Thus, the first player has a winning strategy for* 95 *values between 1 and 100.*

9. There exists a unique pair of positive integers k, n such that k is divisible by 6, and $\sum_{i=1}^{k} i^2 = n^2$. Find (k, n).

Solution. *The sum evaluates to* $\frac{k(k+1)(2k+1)}{6}$. *Since* $k, k+1, 2k+1$ *are pairwise relatively prime, so are the integers* $\frac{k}{6}, k+1, 2k+1$, *and since their product is a perfect square, each itself is a perfect square. Since* $k = (2k+1) - (k+1)$, *and* $k+1$ *and* $2k+1$ *have the same parity,* k *is either odd or divisible by* 4. *But we are given that* k *is divisible by* 6, *so* k *is not odd and is thus divisible by* 4. *Since it's divisible by both* 4 *and* 6, *it's divisible by* 12, *so say* $k = 12m'$. *Then* $k/6 = 2m'$ *is a perfect square, so* m *is even, and* k *is in fact divisible by* 24. *So we have* $k = 24m$, *and all of* $4m, 24m+1, 48m+1$ *are perfect squares. If we make the obvious choice of* $m = 1$, *we get* $4, 25, 49$, *all of which are perfect squares, and so* $k = 4 \cdot 6 = 24$, *and* $n^2 = 4 \cdot 25 \cdot 49$, *and so* $n = 2 \cdot 5 \cdot 7 = 70$. *Thus, the answer is* $(24, 70)$.

10. A partition of a positive integer n is a summing $n_1 + ... + n_k = n$, where $n_1 \geq n_2 \geq ... \geq n_k$. Call a partition *perfect* if every $m \leq n$ can be represented uniquely as a sum of some subset of the n_i's. How many perfect partitions are there of $n = 307$?

Solution. *We claim that the number of partitions of* n *is equivalent to the number of ordered factorizations of* $n+1$. *Note that since 1 needs to be represented,* $n_k = 1$. *Suppose we have* k_1 *1's in our partition. Then* $k_1 + 1$ *must be the next largest number in the partition. Similarly, the next largest must be* $(k_1+1)(k_2+1)$, *where we have* k_2 *repetitions of* $k_1 + 1$. *This continues, until we have* k_i *repetitions of each* $(k_1+1)...(k_{i-1}+1)$ *for all* i *from 1 to* a *for some* a. *Lastly, we have* $k_1 \cdot 1 + k_2(k_1+1) + ... + k_a(k_1+1)...(k_{a-1}+1) = n$, *which simplifies to* $(k_1+1)...(k_a+1) = n+1$. *Thus, the number of choices we have for* $k_1, ..., k_a$ *is equal to the number of ordered factorings of* $n+1$. *For* $n = 307$, *we have* $308 = 2 \cdot 3 \cdot 53$, *which gives us* 13 *possible ordered factorings.*

P1. Find two disjoint sets N_1 and N_2 with $N_1 \cup N_2 = \mathbb{N}$, so that neither set contains an infinite arithmetic progression.

Solution. *One possible answer is to give* $N_1 = \{1\} \cup \{4, 5, 6\} \cup \{11, 12, 13, 14, 15\} \cup ...$ *and* $N_2 = \{2, 3\} \cup \{7, 8, 9, 10\} \cup ...$
It is clear here that we cannot have an arithmetic progression in either set, because both of them contain arbitrarily large intervals with no elements of the set in them.

- *2 points for writing a correct distribution into 2 sets*
- *4 points for correct justification*

P2. Suppose $k > 3$ is a divisor of $2^p + 1$, where p is prime. Prove that $k \geq 2p + 1$.

Solution. *WLG assume* k *is prime. We have* $2^p + 1 \equiv 0 \mod k$, *and thus* $2^p \equiv -1 \mod k$. *This means* $2^{2p} \equiv 1 \mod k$, *and we see that* $2p$ *is the order of 2 mod* k. *Now,* $2^{k-1} \equiv 1$ *by FLT. so* $2p | k-1$ *and so* $k \geq 2P + 1$.

- *1 point for looking at the problem modulo* k
- *1 point for mentioning* $2^{k-1} \equiv 1$ *by FLT*
- *2 points for proving* $2p$ *is the order of 2 mod* k
- *2 pts for full solution*