Time limit: 60 minutes.
Maximum score: 70 points.
Instructions: For this test, you work in teams of six to solve a multi-part, proof-oriented question.

Problems that use the words "compute", "list", or "draw" only call for an answer; no explanation or proof is needed. Unless otherwise stated, all other questions require explanation or proof. Answers should be written on sheets of scratch paper, clearly labeled, with every problem *on its own sheet*. If you have multiple pages for a problem, number them and write the total number of pages for the problem (e.g. 1/2, 2/2).

Write your team ID number and team name clearly on each sheet. Only submit one set of solutions for the team. Do not turn in any scratch work. After the test, put the sheets you want graded into your team packet in order by problem number. If you do not have your packet, ensure your sheets are labeled *extremely clearly* and stack the loose sheets neatly.

In your solution for a given problem, you may cite the statements of earlier problems (but not later ones) without additional justification, even if you haven't solved them.

The problems are ordered by content, NOT DIFFICULTY. It is to your advantage to attempt problems from throughout the test.

No calculators.

Introduction

In this round, we will be explicitly focused on analyzing a function known as the **arithmetic derivative**. While the name is rooted in calculus, this function is purely a number theoretic phenomena. Nevertheless, it shares some interesting mathematical characteristics with its calculus counterpart. In this round, we will be looking at the function definition, examine bounds on the function, and use the function definition to solve arithmetic differential equations. The arithmetic derivative has also been shown to give insight into abstract number theory problems.

Definition. The **Arithmetic Derivative**, $D : \mathbb{Z}^+ \to \mathbb{Z}^+$ is defined as follows:

- $D(0) = 0$
- $D(p) = 1$ if p is prime
- *Product Rule:* If $n = ab$ for $a, b \in \mathbb{N}$, then $D(n) = aD(b) + bD(a)$.

Example. For example, if we wanted to calculate the arithmetic derivative of 10, we would see that $10 = 5 \cdot 2$. So $D(10) = 5D(2) + 2D(5)$ which is $5 \cdot 1 + 2 \cdot 1 = 7$.

Since this function deals with primes, let's start our analysis of this function by looking at prime powers.

A Note on Proofs

Many of these proofs require the use of induction, a proof technique that is very common in number theory. To prove a statement P that depends on a variable n in the natural numbers (i.e 1, 2, 3 ...) is true using induction, first prove that P is true for 1. Then show that if P is true for a natural number k, then P is true for the natural number $k - 1$. An example is illustrated below.

Example. Prove that $\sum_{i=1}^{n} 2 \cdot i - 1 = n^2$. In other words show that the sum of the first n odd numbers is n^2.

Proof. We proceed using induction. Our statement $P(n)$ is "the sum of the first n odd numbers is n^2" First we will prove the base case or $P(1)$. In effect this means we need to show that the sum of the first odd number is 1^2, which is clearly true since $1 = 1$.

Now, we will show that if the statement "the sum of the first n odd numbers is n^2" is true for $n = k$, then it is true for $n = k + 1$. So we assume that the sum of the first k odd numbers is k^2. So we have $\sum_{i=1}^{k} 2 \cdot i - 1 = k^2$. Adding the $k+1$st odd number or $2(k+1) - 1 = 2k + 1$ to the sum, we see that $\sum_{i=0}^{k+1} 2 \cdot i - 1 = k^2 + 2k + 1 = (k + 1)^2$. Note that this is exactly in the same thing as saying "the sum of the first $k + 1$ odd numbers is $(k + 1)^2$, so we have shown that $P(k + 1)$ is true given $P(k)$ is true.

Therefore, since we have shown that $P(1)$ is true and that $P(k)$ implies $P(k+1)$, we have shown that the statement P is true on the natural numbers.

□

1. **[2 pts]** Prove that $\sum_{i=1}^{n} i^2 = \frac{n(2n+1)(n+1)}{6}$.

The Arithmetic Derivative on Prime Powers

2. **[1 pt each]** Find the arithmetic derivative of the following prime powers. You must show your work.

   (a) 5^3

   (b) 11^5

   (c) Find and prove a formula for the arithmetic derivative of p^4 for any prime p.

In general, we can express the arithmetic derivative of prime powers using what is known as the *power rule*.

Theorem. *Suppose natural number $n = p^k$ for some prime p and nonnegative integer k. Then, $D(p^k) = kp^{k-1}$.*

3. Let's try to prove the above theorem true.

   (a) **[1 pt]** Show that $D(1) = 0$.

   (b) **[4 pts]** Prove the above theorem. *Hint:* Use induction.

The Arithmetic Derivative On All Integers

4. Let us compute the arithmetic derivative on general numbers.

   (a) **[1 pt]** Compute $D(899)$

   (b) **[1 pt]** Compute $D(36)$

   (c) **[1 pt]** We saw that the power rule states that $D(p^k) = kp^{k-1}$ if p is a prime. Give a counter-example to show that this does not hold for a general n. In other words show that $D(n^k) = kn^{k-1}$ does not hold.

5. **[6 pts]** Let $n = p_1^{e_1} p_2^{e_2} ... p_k^{e_k}$ be the prime factorization of n. Prove that $D(n) = n \sum_{i=1}^{k} \frac{e_i}{p_i}$.

6. Now that we have proved a general formula, let's go back and reexamine the power rule.

   (a) **[1 pt]** Compute $D(720)$

   (b) **[1 pt]** Compute $D(12^3)$

   (c) **[1 pt]** Compute $D(14^5)$. (*Hint*: $14^4 = 38416$).

   (d) **[3 pts]** Find and prove a formula for $D(n^k)$ in terms of $D(n)$, n, and k.

7. **The Goldbach Conjecture** is a famous conjecture in mathematics that states that for any even integer $2k > 2$, there exists two primes p, q such that $p + q = 2k$.

   [4 pt] Consider the equation $D(n) = 2k$. Show that if there exists a $k \in \mathbb{N}$, greater than 1 such that $D(n) \neq 2k$, for all $n \in \mathbb{N}$, then the Goldbach Conjecture is false.

Higher Order Arithmetic Derivatives

We have explored the notion of the arithmetic derivative. Now, let us see what happens as we iterate this function.

Definition. Define the kth order derivative, $D^{(k)}(n)$ of a natural number n as:

$$\begin{cases} D(n) & \text{if } k = 1. \\ D(D^{(k-1)}(n)) & \text{if } k > 1 \end{cases}$$

Example. As an example, we will compute $D^{(3)}(21)$.

We see that $D^{(3)}(21) = D(D(D(21)))$. $D(21) = 21 \cdot (\frac{1}{7} + \frac{1}{3}) = 10$. $D(10) = 10(\frac{1}{5} + \frac{1}{2}) = 7$. Finally since 7 is prime, $D(7) = 1$, and we are done.

8. Compute the following arithmetic derivatives.

   (a) **[1 pt]** $D^{(2)}(34)$
   (b) **[1 pt]** $D^{(3)}(49)$
   (c) **[1 pt]** $D^{(4)}(3125)$
   (d) **[1 pt]** $D^{(4)}(64)$

9. In this problem, we examine the case in which $D(n) = n$. This is known as an *arithmetic differential equation.*

   (a) **[2 pt]** Show that if $n = p^p$ for a prime p, then $D(n) = n$.
   (b) **[5 pts]** Prove that the numbers $n = p^p$ for any prime p are the only solutions to $D(n) = n$.

Sequences of Higher Order Derivatives

Let us now look at sequence of derivatives. That is, we will consider the sequence of the higher order derivatives of n as $\Delta(n)$. Specifically, we will define $(\Delta(n))_k$ as the sequence

$$D^{(1)}(n), D^{(2)}(n), D^{(3)}(n), D^{(4)}(n) \ldots$$

where the kth term in the sequence is $D^{(k)}(n)$.

Definition. Sequences like these can either be increasing, decreasing, or neither. We call a sequence $(s)_n$ **increasing** if $s_{n+1} \geq s_n$ for all $n \in \mathbb{N}$. A sequence is **decreasing** if $s_{n+1} \leq s_n$ for all $n \in \mathbb{N}$.

10. (a) **[1 pt]** Is the sequence, $(\Delta(12))$, increasing, decreasing, or neither? You do not need to justify your answer.
    (b) **[1 pt]** Is the sequence, $(\Delta(14))$, increasing, decreasing, or neither? You do not need to justify your answer.
    (c) **[2 pts]** Find an example of a number such that $(\Delta(n))$ is neither increasing nor decreasing. You must prove your answer.
    (d) **[2 pts]** For which k is the sequence $(\Delta(2^k))$, increasing? Your answer should be a condition on k. You must prove your answer.

11. **[3 pts]** Show that if $n = k \cdot p^p$ for some natural number $k > 1$ and prime p, then $(\Delta(n))$ is *strictly increasing*, meaning $s_{n+1} > s_n$ for all n.

12. **[7 pts]** Suppose $(\Delta(n))$ is such that it alternates between two distinct numbers m and n. Show that $gcd(n, m) = 1$ and neither m nor n are divisible by a square number other than 1.

Bounds on the Arithmetic Derivative

Let's try to understand what bounds the arithmetic derivative.

13. **[4 pts]** Let p^* be the smallest prime factor of n. Show that $D(n) \leq \frac{n \log_{p^*}(n)}{p^*}$.

14. **[4 pts]** Let k be the sum of all the exponent values in the prime factorization of n. Show that $D(n) \geq k \cdot n^{1-\frac{1}{k}}$

15. **[6 pts]** Let k be the sum of all the exponent values in the prime factorization of n. Show that $D(n) \leq \frac{k-1}{2}n + 2^{k-1}$