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1. Let p be a polynomial with degree less than 4 such that $p(x)$ attains a maximum at $x = 1$. If $p(1) = p(2) = 5$, find $p(10)$.

   Answer: 5

   Solution: Since p attains a maximum at $x = 1$, we know that either p is a parabola or p is a constant polynomial. Since $p(1) = p(2) = 5$, p must be constant.

2. Let A, B, C be unique collinear points $AB = BC = \frac{1}{3}$. Let P be a point that lies on the circle centered at B with radius $\frac{1}{3}$ and the circle centered at C with radius $\frac{1}{3}$. Find the measure of angle PAC in degrees.

   Answer: 30 (degrees)

   Solution: The angle PCB has measure 60 degrees since $PC = PB = BC$. By the inscribed angle theorem, the measure of angle PAC is one half of that, which is $\boxed{30}$ degrees.

3. If $f(x+y) = f(xy)$ for all real numbers x and y, and $f(2019) = 17$, what is the value of $f(17)$?

   Answer: 17

   Solution: If we set $x = 0$, then we find that $f(0 + y) = f(0 \cdot y)$, or $f(y) = f(0)$ for all real y. In other words, f is a constant function; thus, $f(17) = \boxed{17}$.

4. Justin is being served two different types of chips, A-chips, and B-chips. If there are 3 B-chips and 5 A-chips, and if Justin randomly grabs 3 chips, what is the probability that none of them are A-chips?

   Answer: $\mathbf{\frac{1}{56}}$

   Solution: If none of the chips Justin grabs are A-chips, then all of them must be B-chips. The probability that all 3 chips are B-chips is $\frac{3}{8} \cdot \frac{2}{7} \cdot \frac{1}{6} = \boxed{\frac{1}{56}}$.

5. Point P is $\sqrt{3}$ units away from plane A. Let Q be a region of A such that every line through P that intersects A in Q intersects A at an angle between 30° and 60°. What is the largest possible area of Q?

   Answer: $\mathbf{8\pi}$

   Solution: Realize that the area is the difference of two circles with radius 1 and radius 3 respectively. The area is just $(3^2 - 1^2)\pi = \boxed{8\pi}$.

6. How many square inches of paint are needed to fully paint a regular 6-sided die with side length 2 inches, except for the $\frac{1}{3}$-inch diameter circular dots marking 1 through 6 (a different number per side)? The paint has negligible thickness, and the circular dots are non-overlapping.

   Answer: $\mathbf{24 - \frac{7\pi}{12}}$ or $\mathbf{\frac{288-7\pi}{12}}$

   Solution: A blank die needs $2 \cdot 2 \cdot 6$ square inches of paint, and the $1 + 2 + 3 + 4 + 5 + 6 = 21$ dots take up $21\pi r^2 = \frac{7\pi}{12}$. Therefore, a total of $\boxed{24 - \frac{7\pi}{12}}$ square inches of paint is needed.

7. Let $\triangle ABC$ be an equilateral triangle with side length M such that points E_1 and E_2 lie on side AB, F_1 and F_2 lie on side BC, and G_1 and G_2 lie on side AC, such that $m = \overline{AE_1} = \overline{BE_2} =$

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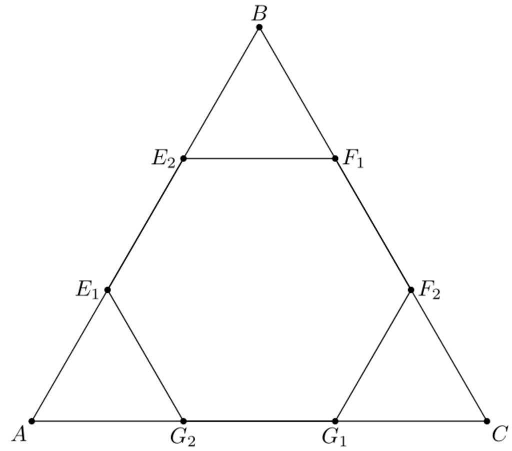
$\overline{BF_1} = \overline{CF_2} = \overline{CG_1} = \overline{AG_2}$ and the area of polygon $E_1E_2F_1F_2G_1G_2$ equals the combined areas of $\triangle AE_1G_2$, $\triangle BF_1E_2$, and $\triangle CG_1F_2$. Find the ratio $\frac{m}{M}$.



Answer: $\mathbf{\frac{1}{\sqrt{6}}}$ **or** $\mathbf{\frac{\sqrt{6}}{6}}$

Solution: The area of an equilateral triangle with side length m is $\frac{m^2\sqrt{3}}{4}$, so the areas of the smaller triangles adds up to $3\frac{m^2\sqrt{3}}{4}$, and the area of the hexagon is $\frac{M^2\sqrt{3}}{4} - 3\frac{m^2\sqrt{3}}{4}$. Equating the two quantities and simplifying, $M^2 = 6m^2$, so $\frac{m}{M} = \boxed{\frac{1}{\sqrt{6}}}$.

8. Let $\varphi = \frac{1}{2019}$. Define

$$g_n = \left\{ \begin{array}{ll} 0 & \text{if round}(n\varphi) = \text{round}\left((n-1)\varphi\right) \\ 1 & \text{otherwise.} \end{array} \right\}$$

where round (x) denotes the round function.
Compute the expected value of g_n if n is an integer chosen from interval $[1, 2019^2]$.

Answer: $\mathbf{\frac{1}{2019}}$

Solution: Since $\left\lfloor \frac{1009}{2019} \right\rceil = 0$ while $\left\lfloor \frac{1010}{2019} \right\rceil = 1$, we can see that $g_n = 1$ only when $n = 1010 + 2019k$ (for integer k), and $g_n = 0$ otherwise. Thus, the expected value of g_n is $\boxed{\frac{1}{2019}}$.

9. Define an *almost-palindrome* as a string of letters that is not a palindrome but can become a palindrome if one of its letters is changed. For example, TRUST is an almost-palindrome because the R can be changed to an S to produce a palindrome, but TRIVIAL is not an almost-palindrome because it cannot be changed into a palindrome by swapping out only one letter (both the A and the L are out of place). How many almost-palindromes contain fewer than 4 letters?

Answer: 17550

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Solution: No almost-palindrome can contain only 1 letter because all 1-letter strings are palindromes. In order for a 2-letter string to be an almost-palindrome, it must contain distinct letters; there are $26 \cdot 25$ of those. The 3-letter strings that are almost-palindromes either have 3 distinct letters or have 2 of the same letter in a row (and 1 other letter tacked on). In the former case, there are $26 \cdot 25 \cdot 24$ almost-palindromes; in the latter, there are $2 \cdot 26 \cdot 25$ (to account for order). Thus, there are a total of $26 \cdot 25 + 26 \cdot 25 \cdot 24 + 2 \cdot 26 \cdot 25 = 27 \cdot 26 \cdot 25 = \boxed{17550}$ almost-palindromes with fewer than 4 letters.

10. Let $MATH$ be a square with $MA = 1$. Point B lies on $\overline{AT}$ such that $m\angle MBT = 3.5m\angle BMT$. What is the area of $\triangle BMT$?

Answer: $\mathbf{\frac{\sqrt{3}-1}{2}}$ **or** $\mathbf{\frac{\sqrt{3}}{2} - \frac{1}{2}}$ **or** $\mathbf{0.5(\sqrt{3}-1)}$ **or** $\mathbf{0.5\sqrt{3} - 0.5}$

Solution: Let $m\angle BMT = a$. Then $m\angle MBA = (180 - 3.5a)$; since $\triangle MAB$ is right, $m\angle AMB = (3.5a - 90)$. Thus, $3.5a - 90 + a = 45$, as $m\angle AMT = 45°$ ($\overline{MT}$ is a diagonal of the rectangle). It follows that $a = 30$, so $m\angle AMB = 15°$; thus, $\triangle MAB$ is a 15-75-90 triangle. Recalling our trig ratios, we obtain $\tan 15° = 2 - \sqrt{3}$, which yields $AB = 2 - \sqrt{3}$. We obtain an area of $\boxed{\frac{\sqrt{3}-1}{2}}$ for $\triangle BMT$ as a result.

11. A regular 17-gon with vertices $V_1, V_2, \ldots, V_{17}$ and sides of length 3 has a point P on $\overline{V_1V_2}$ such that $\overline{V_1P} = 1$. A chord that stretches from V_1 to V_2 containing P is rotated within the interior of the heptendecagon around V_2 such that the chord now stretches from V_2 to V_3. The chord then hinges around V_3, then V_4, and so on, continuing until P is back at its original position. Find the total length traced by P.

Answer: $\mathbf{45\pi}$

Solution: Note that the end trace is a set of circle-arcs of radius 1 and 2 each with total angle equal to the sum of internal angles of the heptendecagon. As a result, the total length is $2 \cdot \pi \cdot 15 + 1 \cdot \pi \cdot 15 = \boxed{45\pi}$/

12. Box is thinking of a number, whose digits are all "1". When he squares the number, the sum of its digit is 85. How many digits is Box's number?

Answer: **11**

Solution: Notice that

$$\begin{aligned} 1 \cdot 1 &= 1 \\ 11 \cdot 11 &= 121 \\ 111 \cdot 111 &= 12321 \\ 1111 \cdot 1111 &= 1234321, \end{aligned}$$

etc., until you get to 10, when it becomes 1234567900987654321. This is because, at the digit where the 10 is supposed to go, it could only take up one spot, so the 1 gets carried to where the 9 goes, which, when added, becomes a 10, which gets carried again. The next digit is an 8, which becomes a 9. Thus, the 8 disappears. Using similar logic, we can calculate that $11 \cdots 1$ with 11 digits has sum 85. Another way to see this is by computing the past powers of 1's and using the fact that the sum of the digits is the same modulo 3. One can eliminate many possibilities and

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that if there are enough 1's the sum grows fast.

Note: Box's number doesn't have to be an integer.

13. Two circles O_1 and O_2 intersect at points A and B. Lines $\overline{AC}$ and $\overline{BD}$ are drawn such that C is on O_1 and D is on O_2 and $\overline{AC} \perp \overline{AB}$ and $\overline{BD} \perp \overline{AB}$. If minor arc $\overset{\frown}{AB}= 45$ degrees relative to O_1 and minor arc $\overset{\frown}{AB}= 60$ degrees relative to O_2 and the radius of $O_2 = 10$, the area of quadrilateral $CADB$ can be expressed in simplest form as $a + b\sqrt{k} + c\sqrt{\ell}$. Compute $a + b + c + k + \ell$.

Answer: 155

Solution: Note that since $\angle CAB$ and $\angle BDA$ are both 90 degrees, BC and AD are diameters of circles O_1 and O_2, respectively. Thus, the area of the quadrilateral is the area of triangle ABD plus the area of triangle ABC. Since arc AB is 60 degrees with respect to O_2, the radius of O_2 is 10, so we can compute that $BD = 10\sqrt{3}$, so the area of triangle ABD is $5 \cdot 10\sqrt{3} = 50\sqrt{3}$. Let r be the radius of circle O_1. Then by the law of cosines,

$$2r^2 - 2r^2 \cdot \frac{\sqrt{2}}{2} = 100$$

$$2r^2 + 2r^2 \cdot \frac{\sqrt{2}}{2} = AC^2 := x^2.$$

Using this, we can say that

$$x^2 = \frac{100(2+\sqrt{2})}{2-\sqrt{2}}$$

so

$$x = 10\sqrt{\frac{2+\sqrt{2}}{2-\sqrt{2}}} = 10 + 10\sqrt{2}.$$

Thus, the area of ABC is $50+50\sqrt{2}$, and the area of quadrilateral $CADB$ is $50+50\sqrt{2}+50\sqrt{3}$, which yields an answer of $\boxed{155}$.

14. On a 24 hour clock, there are two times after 01:00 for which the time expressed in the form hh:mm and in minutes are both perfect squares. One of these times is 01:21, since 121 and 60+21 = 81 are both perfect squares. Find the other time, expressed in the form hh:mm.

Answer: 20:25

Solution: We can guess and check to find that $\boxed{20:25}$ and $20 \cdot 60 + 25 = 1225$ are both perfect squares, so that solution works. Now let us prove these are the only two solutions.

We wish to find h, m, x, and y for which $100h + m = x^2$ and $60h + m = y^2$. We find $h = (x^2 - y^2)/40$, so $x^2 - y^2$ is divisible by both 8 and 5. The first implies that $x \equiv y \pmod 2$, whereas the second implies $x \equiv \pm y \pmod 5$. This means either $x = y + 10k$ or $x = -y + 10k$.

In the first case, we have $h = \dfrac{k(y+5k)}{2}$ and $m = y^2 - 30ky - 150k^2$. Since $m \geq 0$, we get $y \geq 30k$. This implies $h \geq \dfrac{35k^2}{2}$, and thanks to the bounds on h, further implies that $k = 1$. Now we know $h = \dfrac{(y+5)}{2}$ and $m = y^2 - 30y - 150$. The first equation tells us that y is odd. Using the second one along with the bound $0 \leq m < 60$ tells us that $y = 35$, which gives the solution $h = 20$ and $m = 25$ from above.

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In the second case, we can write $x = 5k+z, y = 5k-z$. This gives $h = \dfrac{kz}{2}, m = z^2-40kz+25k^2 = z^2 + 5k(5k - 8z)$. By enforcing our bounds, we find that $k > z$ must hold. If $k = z + 1$, $b = z^2 + 5(z + 1)(5 - 3z)$, which is negative for $z \geq 2$. If $z = 1$, $k = 2$ gives $b = 21$, $a = 1$ which is the solution provided in the problem statement. Now we know $k > z + 1$ and kz is even and less than 48, which gives us finitely many cases to check. Furthermore, for a fixed z, once we've found a k giving an m thats too big, we can check the next z. If $z = 2$, $k = 4$ gives $m = 84$, which is too big. If $z = 3$, $k = 6$ gives $m = 189$, again too big. If $z = 4$, $k = 6$ gives -44, $k = 7$ gives $b = 119$, too big. If $z = 5$, $k = 8$ gives $h = 25$, the solution we just found above. Those are all the possibilities for which $kz < 48$, as desired.

15. How many distinct positive integers can be formed by choosing their digits from the string 04072019?

Answer: 12340

Solution:

Suppose the number of digits is t. Suppose there are r non-zero digits. Then, total number of distinct numbers with t digits and r nonzero digits is $\frac{5!}{(5-r)!} \cdot \binom{t-1}{t-r}$. The first term comes from the fact that we need to pick r non-zero digits out of 5 total possible with ordering. The second term is how we arrange the $t - r$ zero digits in the last $t - 1$ places (since 0 cannot occupy the first position). In the case there is only one digit, we can disregard the second term.

So if $t = 1$, we know that $r = 1$ and so our only term is $\frac{5!}{4!} = 5$. If $t = 2$, then we can have $r = 1$ or 2, from which we get 5 and 20. If $t = 3$ then $r = 1, 2, 3$, from which we get $5, 40, 60$. If $t = 4$ we have $r = 1, 2, 3, 4$ from which we get $5, 60, 180, 120$. When $t = 5$, we have $r = 2, 3, 4, 5$ since we know that the number of zero digits must be less than or equal to 3 so $t - r \leq 3$. In this case we get $80, 360, 480, 120$. When $t = 6$ we get $r = 3, 4, 5$ (since $r \leq 5$) which is $600, 1200, 600$. At $t = 7$ we have $r = 4, 5$ with 1800 and 2400. And at $t = 8$ we have $r = 5$ and 4200. Adding all of these numbers up, we come up with our final answer of $\boxed{12340}$.

16. Let ABC be a triangle with $AB = 26, BC = 51$, and $CA = 73$, and let O be an arbitrary point in the interior of $\triangle ABC$. Lines l_1, l_2, and l_3 pass through O and are parallel to $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$, respectively. The intersections of l_1, l_2, and l_3 and the sides of $\triangle ABC$ form a hexagon whose area is A. Compute the minimum value of A.

Answer: 280

Solution: First realize that the minimum area of the hexagon is $\dfrac{2}{3}$ that of the triangle, regardless of the angle measures of the triangle. This can be proven using arguments regarding parallel lines and similar triangles. Proof:

Notation: l_1 intersects AC at D, BC at G; l_2 intersects AB at F, AC at I; l_3 intersects AB at E, BC at H. Then $DEFGHI$ is the hexgon whose minimum area is wanted.

By tracing parallel lines and similar triangles, we have

$$\begin{aligned}
\frac{A_{EFO}}{A_{ABC}} + \frac{A_{OGH}}{A_{ABC}} + \frac{A_{DOI}}{A_{ABC}} &= \left(\frac{FO}{BC}\right)^2 + \left(\frac{GH}{BC}\right)^2 + \left(\frac{OI}{BC}\right)^2 \\
&= \left(\frac{BG}{BC}\right)^2 + \left(\frac{GH}{BC}\right)^2 + \left(\frac{HC}{BC}\right)^2 \\
&\geq \frac{1}{3}\left(\frac{BG}{BC} + \frac{GH}{BC} + \frac{HC}{BC}\right)^2 = \frac{1}{3}
\end{aligned}$$

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Where the last line follows from Cauchy-Schwarz inequality. Hence we have $A_{EFO} + A_{OGH} + A_{DOI} \geq \frac{1}{3}A_{ABC}$, and from there $A_{AED} + A_{FBG} + A_{IHC} \leq \frac{1}{2}(1 - \frac{1}{3})A_{ABC} = \frac{1}{3}A_{ABC}$. Therefore, the area of the hexgon $A_{DEFGHI} = A_{ABC} - (A_{AED} + A_{FBG} + A_{IHC}) \geq 1 - \frac{1}{3} = \frac{2}{3}$. The equality is reached when O is the centroid of triangle ABC.

Use Heron's formula and we can get the area of the triangle is $\sqrt{75 \cdot 49 \cdot 24 \cdot 2} = 420$. Thus the minimum area of the hexagon is $\dfrac{420}{3} \cdot 2 = \boxed{280}$

17. Let C be a circle of radius 1 and O its center. Let $\overline{AB}$ be a chord of the circle and D a point on $\overline{AB}$ such that $OD = \dfrac{\sqrt{2}}{2}$ such that D is closer to A than it is to B, and if the perpendicular line at D with respect to $\overline{AB}$ intersects the circle at E and F, $AD = DE$. The area of the region of the circle enclosed by $\overline{AD}$, $\overline{DE}$, and the minor arc AE may be expressed as $\dfrac{a + b\sqrt{c} + d\pi}{e}$ where a, b, c, d, e are integers, $\gcd(a, b, d, e) = 1$, and c is squarefree. Find $a + b + c + d + e$.

Answer: 16

Solution: Reflect lines $\overline{AC}$ and $\overline{EF}$ across the origin. This forms a square in the center with 8 other regions, four of which are the same area as the desired region, and four of which are the same area. Call our desired area a and the other area b. Then we obtain $4(a + b) = \pi - 1$. Looking at the sector cut off by $\overline{AB}$, we may compute that the angle of that is $120°$; thus, it has area $\dfrac{\pi}{3}$. Triangle OAB has area $\dfrac{\sqrt{3}}{4}$. Thus, we obtain $2a + b = \dfrac{\pi}{3} - \dfrac{\sqrt{3}}{4}$. As a result, we find that

$$a = \frac{\pi}{3} - \frac{\sqrt{3}}{4} - \frac{\pi}{4} + \frac{1}{4} = \frac{\pi}{12} - \frac{\sqrt{3} - 1}{4},$$

which yields an answer of $\boxed{16}$.

18. Define $f(x, y)$ to be $\dfrac{|x|}{|y|}$ if that value is a positive integer, $\dfrac{|y|}{|x|}$ if that value is a positive integer, and zero otherwise.

We say that a sequence of integers l_1 through l_n is *good* if $f(l_i, l_{i+1})$ is nonzero for all i where $1 \leq i \leq n - 1$, and the score of the sequence is $\displaystyle\sum_{i=1}^{n-1} f(l_i, l_{i+1})$.

Compute the maximum possible score of a good subsequence subject to the further constraints that the absolute value of every element is between 2 and 6, and that if b directly follows a in the sequence, it can only do so once, and a cannot directly follow b afterwards.

Answer: 37

Solution: To maximize the score of the sequence, we want to maximize the sum of ordered pairs $f(l_i, l_{i+1})$ while maintaining its goodness. If we start with 6, then we can continue through the factors of 6 (positive and negative) and then make sure not to repeat values in accordance with the stipulations. Once we reach 6 again, we are done. In this manner, the sequence is 6, 6, 3, 3, −6, -6, −3, -3, 6, 6, 2, 2, 4, 4, −2, -2, −4, -4, 2, 2, −6, -6, −2, -2, 6, 6 so the maximum score is $\boxed{37}$. (This order in particular is optimal, since we are going through the largest numbers that have the smallest number of factors first - namely 3, which is the largest prime factor of 6.) Note that, if we were to start with a value other than 6, then that number would have fewer factors than 6, and so would result in $f(x, y) = 0$ sooner than would 6 (as we would exhaust the set of possible factors; if we increase the absolute value between two numbers, then the likelihood

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of having $f(x, y) = 0$ increases, as clearly a smaller number cannot be a multiple of the larger number). Hence no other solutions can be optimal.

19. Let a and b be real numbers such that

$$\max_{0\le x\le 1} |x^3 - ax - b|$$

is as small as possible. Find $a + b$ in simplest radical form. (Hint: If $f(x) = x^3 - cx - d$, then the maximum (or minimum) of $f(x)$ either occurs when $x = 0$ and/or $x = 1$ and/or when x satisfies $3x^2 - c = 0$).

Answer: $\mathbf{1 - \frac{\sqrt{3}}{9}}$ **or** $\mathbf{\frac{9-\sqrt{3}}{9}}$

Solution: Let's first take an intuitive guess at what the answer to this question is. First, consider $x^3 - x$. This has a minimum in $[0, 1]$ at $\left(\frac{\sqrt{3}}{3}, -\frac{2\sqrt{3}}{9}\right)$. Shifting the graph up by $\frac{\sqrt{3}}{9}$, we guess that $x^3 - x + \frac{\sqrt{3}}{9}$ achieves the minimum. Define a function $\ell : \mathbb{R}[x] \to \mathbb{R}$ (where $\mathbb{R}[x]$ is the set of polynomials of x with coefficients in $\mathbb{R}$) such that

$$\ell(f) = \frac{\frac{1}{\sqrt{3}}f(1) + (1 - \frac{1}{\sqrt{3}})f(0) - f(\frac{\sqrt{3}}{3})}{2}.$$

Since $\dfrac{|1| + \left|\frac{1}{\sqrt{3}}\right| + \left|1 - \frac{1}{\sqrt{3}}\right|}{2} = 1$, we have $\ell(f) \le \max_{0\le x\le 1} |f(x)|$. In addition, $\ell\left(x^3 - x + \frac{\sqrt{3}}{9}\right) = \frac{\sqrt{3}}{9} = \max_{0\le x\le 1} |p(x)|$ where $p(x) = x^3 - x + \frac{\sqrt{3}}{9}$, and $\ell(ax + b) = 0$ for real a and b. If some other $cx+d$ achieves a lower value than $x - \frac{\sqrt{3}}{9}$ for $|x^3 - cx - d|$, then $\ell(x^3 - cx - d) > \max_{0\le x\le 1} |x^3 - cx - d|$.

20. Define a sequence F_n such that $F_1 = 1$, $F_2 = x$, $F_{n+1} = xF_n + yF_{n-1}$ where and x and y are positive integers. Suppose

$$\frac{1}{F_k} = \sum_{n=1}^{\infty} \frac{F_n}{d^n}$$

has exactly two solutions (d, k) with $d > 0$ is a positive integer. Find the least possible positive value of d.

Answer: 3

Solution: Let $F_1 = a$, $F_2 = b = x$. Notice that $F_{n+1} = xF_n + yF_{n-1}$ and that $b = ax$. We begin the solution with a series of steps:

1. First, we give a formula for the sum. Let S denote the sum. Then we have

$$\begin{aligned} S = \sum_{n=1}^{\infty} \frac{F_n}{d^n} &= \frac{a}{d} + \frac{b}{d^2} + \sum_{n=3}^{\infty} \frac{F_n}{d^n} \\ &= \frac{a}{d} + \frac{b}{d^2} + \frac{x}{d}\sum_{n=2}^{\infty} \frac{F_n}{d^n} + \frac{y}{d^2}\sum_{n=1}^{\infty} \frac{F_n}{d^n} \\ &= \frac{a}{d} + \frac{b}{d^2} + \frac{x}{d}\left(S - \frac{a}{d}\right) + \frac{y}{d^2}S \end{aligned}$$

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This implies that

$$S = \frac{ad + b - ax}{d^2 - dx - y} = \frac{1}{F_k}.$$

2. Next, rearrange into a quadratic equation in d:

$$d^2 - d(aF_k + x) - F_k b + aF_k x - y = 0.$$

Since d must be an integer, it must have a square discriminant:

$$(aF_k + x)^2 - 4(-F_k b + axF_k - y) = (aF_k + x)^2 + 4y = z^2$$

for some integer z.

3. Next, we show that $k \leq 2$. The key observation here is that if $x^2 + c = y^2$ with $x, y, c > 0$ are integers, then $c \geq 2x + 1$. If $k \geq 4$, then $F_k \geq 1 + 2y$ (the minimum occurs when $F_1 = F_2 = 1$ and $x = 1$), so $2F_k > 4y$, so $(aF_k + x)^2 + 4y$ cannot possibly be a perfect square. Furthermore, if $d = 3$, then $a = 1$, since if $a \geq 2$, then $2 \cdot (aF_k + x) \geq 2(2(1 + y)) > 4y$. If $a = 1$, then $F_3 = bx + y$. Notice in addition that $(y + 2)^2 - y^2 = 4y + 4$, so $aF_k + x = bx + y + x \leq y + 1$, so $bx + x = 1$. This contradicts b, x being positive.

4. Plugging in $F_1 = a$, $F_2 = b$ into the equation, we see that $(ab + x)^2 + 4y = (a^2x + x)^2 + 4y$ and $(a^2 + x)^2 + 4y$ must be perfect squares. Plugging in $a = 1$, we get:

$$4x^2 + 4y = z_1^2$$

$$(x + 1)^2 + 4y = z_2^2.$$

Plugging in $x = 1$, we obtain that $d = 3$ with $k = 1, 2$ are two solutions. But the point is that there is only one d, and hence $d = \boxed{3}$ is the answer.

Remark The intended wording was to have two solutions d for a fixed k. In this case, $x = 1$ doesn't work since we obtain only one solution for d. We see that $x = 2$ yields no value of d, $x = 3$ yields no value, $x = 4$ yields $(z_1, z_2) = (20, 19)$ which gives $d = 12$, $x = 5$ yields no solution, $x = 6$ yields $d = 27$, $x = 7$ yields no such d, $x = 8$ yields $d = 12, 48$, $x = 9$ yields $d = 18$, and finally $x = 10$ yields $d = 28, 75$, and we see that further d cannot yield larger values.