Time limit: 90 minutes.
Instructions: This test contains 20 short answer questions. All answers must be expressed in simplest form unless specified otherwise. Only answers written inside the boxes on the answer sheet will be considered for grading.
No calculators.

1. Let p be a polynomial with degree less than 4 such that $p(x)$ attains a maximum at $x = 1$. If $p(1) = p(2) = 5$, find $p(10)$.

2. Let A, B, C be unique collinear points $AB = BC = \frac{1}{3}$. Let P be a point that lies on the circle centered at B with radius $\frac{1}{3}$ and the circle centered at C with radius $\frac{1}{3}$. Find the measure of angle PAC in degrees.

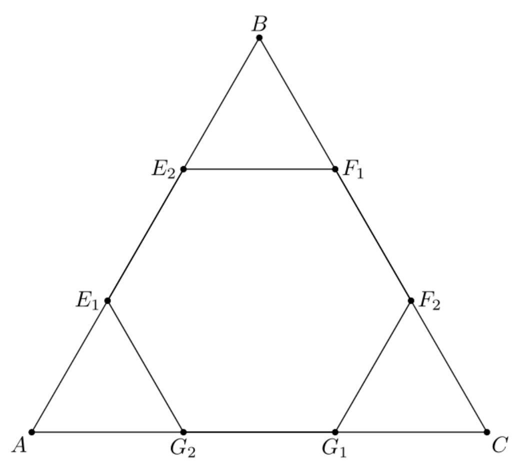
3. If $f(x+y) = f(xy)$ for all real numbers x and y, and $f(2019) = 17$, what is the value of $f(17)$?

4. Justin is being served two different types of chips, A-chips, and B-chips. If there are 3 B-chips and 5 A-chips, and if Justin randomly grabs 3 chips, what is the probability that none of them are A-chips?

5. Point P is $\sqrt{3}$ units away from plane A. Let Q be a region of A such that every line through P that intersects A in Q intersects A at an angle between 30° and 60°. What is the largest possible area of Q?

6. How many square inches of paint are needed to fully paint a regular 6-sided die with side length 2 inches, except for the $\frac{1}{3}$-inch diameter circular dots marking 1 through 6 (a different number per side)? The paint has negligible thickness, and the circular dots are non-overlapping.

7. Let $\triangle ABC$ be an equilateral triangle with side length M such that points E_1 and E_2 lie on side AB, F_1 and F_2 lie on side BC, and G_1 and G_2 lie on side AC, such that $m = \overline{AE_1} = \overline{BE_2} = \overline{BF_1} = \overline{CF_2} = \overline{CG_1} = \overline{AG_2}$ and the area of polygon $E_1E_2F_1F_2G_1G_2$ equals the combined areas of $\triangle AE_1G_2$, $\triangle BF_1E_2$, and $\triangle CG_1F_2$. Find the ratio $\frac{m}{M}$.



8. Let $\varphi = \frac{1}{2019}$. Define

$$g_n = \left\{ \begin{array}{ll} 0 & \text{if } \mathrm{round}(n\varphi) = \mathrm{round}\left((n-1)\varphi\right) \\ 1 & \text{otherwise.} \end{array} \right\}$$

where $\text{round}(x)$ denotes the round function.
Compute the expected value of g_n if n is an integer chosen from interval $[1, 2019^2]$.

9. Define an *almost-palindrome* as a string of letters that is not a palindrome but can become a palindrome if one of its letters is changed. For example, TRUST is an almost-palindrome because the R can be changed to an S to produce a palindrome, but TRIVIAL is not an almost-palindrome because it cannot be changed into a palindrome by swapping out only one letter (both the A and the L are out of place). How many almost-palindromes contain fewer than 4 letters?

10. Let $MATH$ be a square with $MA = 1$. Point B lies on $\overline{AT}$ such that $m\angle MBT = 3.5m\angle BMT$. What is the area of $\triangle BMT$?

11. A regular 17-gon with vertices $V_1, V_2, \ldots, V_{17}$ and sides of length 3 has a point P on $\overline{V_1V_2}$ such that $\overline{V_1P} = 1$. A chord that stretches from V_1 to V_2 containing P is rotated within the interior of the heptendecagon around V_2 such that the chord now stretches from V_2 to V_3. The chord then hinges around V_3, then V_4, and so on, continuing until P is back at its original position. Find the total length traced by P.

12. Box is thinking of a number, whose digits are all "1". When he squares the number, the sum of its digit is 85. How many digits is Box's number?

13. Two circles O_1 and O_2 intersect at points A and B. Lines $\overline{AC}$ and $\overline{BD}$ are drawn such that C is on O_1 and D is on O_2 and $\overline{AC} \perp \overline{AB}$ and $\overline{BD} \perp \overline{AB}$. If minor arc $\overset{\frown}{AB} = 45$ degrees relative to O_1 and minor arc $\overset{\frown}{AB} = 60$ degrees relative to O_2 and the radius of $O_2 = 10$, the area of quadrilateral $CADB$ can be expressed in simplest form as $a + b\sqrt{k} + c\sqrt{\ell}$. Compute $a + b + c + k + \ell$.

14. On a 24 hour clock, there are two times after 01:00 for which the time expressed in the form hh:mm and in minutes are both perfect squares. One of these times is 01:21, since 121 and 60+21 = 81 are both perfect squares. Find the other time, expressed in the form hh:mm.

15. How many distinct positive integers can be formed by choosing their digits from the string 04072019?

16. Let ABC be a triangle with $AB = 26, BC = 51$, and $CA = 73$, and let O be an arbitrary point in the interior of $\triangle ABC$. Lines l_1, l_2, and l_3 pass through O and are parallel to $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$, respectively. The intersections of l_1, l_2, and l_3 and the sides of $\triangle ABC$ form a hexagon whose area is A. Compute the minimum value of A.

17. Let C be a circle of radius 1 and O its center. Let $\overline{AB}$ be a chord of the circle and D a point on $\overline{AB}$ such that $OD = \frac{\sqrt{2}}{2}$ such that D is closer to A than it is to B, and if the perpendicular line at D with respect to $\overline{AB}$ intersects the circle at E and F, $AD = DE$. The area of the region of the circle enclosed by $\overline{AD}$, $\overline{DE}$, and the minor arc AE may be expressed as $\frac{a + b\sqrt{c} + d\pi}{e}$ where a, b, c, d, e are integers, $\gcd(a, b, d, e) = 1$, and c is squarefree. Find $a + b + c + d + e$.

18. Define $f(x, y)$ to be $\frac{|x|}{|y|}$ if that value is a positive integer, $\frac{|y|}{|x|}$ if that value is a positive integer, and zero otherwise.

We say that a sequence of integers l_1 through l_n is *good* if $f(l_i, l_{i+1})$ is nonzero for all i where $1 \leq i \leq n-1$, and the score of the sequence is $\sum_{i=1}^{n-1} f(l_i, l_{i+1})$.

Compute the maximum possible score of a good subsequence subject to the further constraints that the absolute value of every element is between 2 and 6, and that if b directly follows a in the sequence, it can only do so once, and a cannot directly follow b afterwards.

19. Let a and b be real numbers such that

$$\max_{0 \leq x \leq 1} |x^3 - ax - b|$$

is as small as possible. Find $a + b$ in simplest radical form. (Hint: If $f(x) = x^3 - cx - d$, then the maximum (or minimum) of $f(x)$ either occurs when $x = 0$ and/or $x = 1$ and/or when x satisfies $3x^2 - c = 0$).

20. Define a sequence F_n such that $F_1 = 1$, $F_2 = x$, $F_{n+1} = xF_n + yF_{n-1}$ where and x and y are positive integers. Suppose

$$\frac{1}{F_k} = \sum_{n=1}^{\infty} \frac{F_n}{d^n}$$

has exactly two solutions (d, k) with $d > 0$ is a positive integer. Find the least possible positive value of d.