1. We inscribe a circle ω in equilateral triangle ABC with radius 1. What is the area of the region inside the triangle but outside the circle?

   Answer: $\mathbf{3\sqrt{3} - \pi}$.

   Solution: Since the radius of ω is 1, we can use 30-60-90 triangles to get that the side length of ABC is $2\sqrt{3}$. Thus since the area of ω is π and the area of ABC is $\sqrt{3}/4 \cdot (2\sqrt{3})^2 = 3\sqrt{3}$, the desired area is $\boxed{3\sqrt{3} - \pi.}$

2. Define the inverse of triangle ABC with respect to a point O in the following way: construct the circumcircle of ABC and construct lines AO, BO, and CO. Let A' be the other intersection of AO and the circumcircle (if AO is tangent, then let $A' = A$). Similarly define B' and C'. Then $A'B'C'$ is the inverse of ABC with respect to O. Compute the area of the inverse of the triangle given in the plane by $A(-6, -21), B(-23, 10), C(16, 23)$ with respect to $O(1, 3)$.

   Answer: 715

   Solution: Observe that O is the circumcenter of ABC. Because of this, our definition of the inverse and some angle chasing show that the inverse of ABC with respect to O is equivalent to rotating ABC 180° about O. Thus the area of the inverse is the same as the area of ABC, which we can find using the shoelace determinant:

$$-\frac{1}{2}\begin{vmatrix} -6 & -21 & 1 \\ -23 & 10 & 1 \\ 16 & 23 & 1 \end{vmatrix} = 715$$

3. We say that a quadrilateral Q is *tangential* if a circle can be inscribed into it, i.e. there exists a circle C that does not meet the vertices of Q, such that it meets each edge at exactly one point. Let N be the number of ways to choose four distinct integers out of $\{1, \dots, 24\}$ so that they form the side lengths of a tangential quadrilateral. Find the largest prime factor of N.

   Answer: 43

   Solution: Note that the sides of a quadrilateral $ABCD$ in which a circle can be inscribed are of the form $AB = a + b$, $BC = b + c$, $CD = c + d$, $DA = d + a$, i.e. $AB + CD = BC + DA$. (insert picture). The converse also holds true: start with any quadrilateral $ABCD$ with the given side lengths; there exists a circle O tangent to AB, BC, CD centered at the intersection of the bisectors of $\angle ABC$ and $\angle BCD$. Suppose O is not tangent to DA. Then draw the line through A tangent to O, and let P be its intersection with CD. Now $ABCP$ is a quadrilateral with a circle inscribed in it, so $AB + CP = BC + PA$. Assume first that P is between C and D. We have $AB + CD = BC + DA$ so $AB + CP + PD = BC + DA$, and thus $AP + PD = DA$. $\therefore P = D$, and O is tangent to DA. If P is not between C and D then D is between C and P, so we get $AB + CD + DP = BC + PA$ and $AB + CD = BC + DA$. Hence $AD + DP = PA$, so again $P = D$ and O is tangent to AD.

   Let $n \in \mathbf{N}$; we shall restrict to the case where n is even in view of our problem. For each integer k, the number of pairs $1 \le x < y \le n$ such that $x + y = k$ is $\min(n - \lfloor (k-1)/2 \rfloor, \lfloor (k-1)/2 \rfloor)$. Thus for $3 \le k \le n + 1$, the number of pairs for each k is $\lfloor (k-1)/2 \rfloor$, so the number of pairs $(x, y), (z, w)$ such that x, y, z, w distinct and $x + y = z + w = k$ is $2\sum_{i=2}^{i=n/2} \binom{i}{2} = \sum_{i=2}^{i=n/2} i(i-1) = n(n+2)(n-2)/24$. From here, we obtain that the largest prime factor is $\boxed{43}$.

   Remark: The claims above regarding the characterization of tangential quadrilaterals are Pitot's Theorem and its converse (due to Steiner, circa. 1846), respectively. The proof given here can be found at `https://brilliant.org/wiki/pitots-theorem/`.