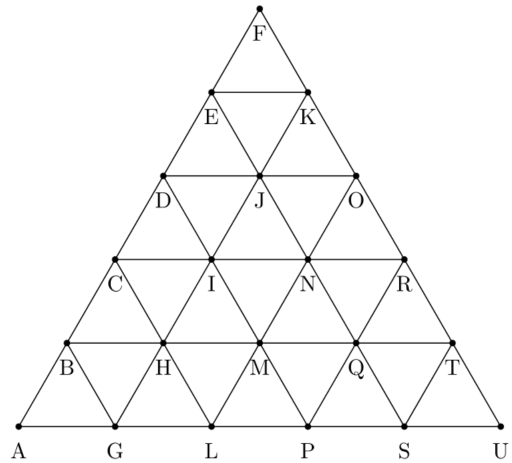
1. Consider the figure below, where every small triangle is equilateral with side length 1. Compute the area of the polygon $AEKS$.



Answer: $5\sqrt{3}$.

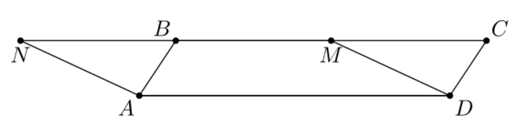
Solution: We see that the figure is a trapezoid and we can calculate via $30-60-90$ triangles that the height of the trapezoid is $2\sqrt{3}$. Using the trapezoid area formula, the area is just $(1+4)/2 \cdot 2\sqrt{3} = \boxed{5\sqrt{3}.}$

2. A set of points in the plane is called full if every triple of points in the set are the vertices of a non-obtuse triangle. What is the largest size of a full set?

Answer: 4

Solution: We claim that 4 is the best we can do. The relevent construction is one where the points are the vertices of a square. Now, it suffices to show that any set of 5 points cannot be full, because any larger set contains a set of 5 points as a subset. Now, if the points form a convex pentagon, then since the average angle between two sides of the pentagon is always $108 > 90$, there must be three points that form an obtuse angle. If the convex hull of the points is a rectangle (if it isn't a rectangle then that subset of 4 points isn't full), then the remaining point is inside the rectangle and thus always induces an obtuse angle with two opposite corners of the rectangle. Finally, if the convex hull of the points is a triangle, the same logic applies. Hence $\boxed{4}$ is the best we can do.

3. Let $ABCD$ be a parallelogram with $BC = 17$. Let M be the midpoint of BC and let N be the point such that $DANM$ is a parallelogram. What is the length of segment NC?



Answer: $\frac{51}{2}$ or 25.5

Solution: Drawing the figure, we see that N, B, M, C are collinear. Moreover, We have $\angle MCD = \angle NBA$ and $\angle NAB = \angle DMC$. Thus, NBA is similar MCD, and since $BA = CD$, they are congruent. Thus, $NB = BM = MC = BC/2$. Hence, the desired length is $3 \cdot 17/2 = \boxed{\frac{51}{2}}$.

4. The area of right triangle ABC is 4, and hypotenuse AB is 12. Compute the perimeter of ABC.

   Answer: $\mathbf{12 + 4\sqrt{10}}$

   Solution: Let $AB = x$ and $BC = y$. Then from the Pythagorean Theorem, we know that $x^2 + y^2 = 144$. Moreover, since the area of ABC is 4, we know that $xy = 8$, hence

$$12 + x + y = 12 + \sqrt{(x+y)^2} = 12 + \sqrt{x^2 + y^2 + 2xy} = 12 + \sqrt{160} = \boxed{12 + 4\sqrt{10}}$$

5. Find the area of the set of all points z in the complex plane that satisfy

$$|z - 3i| + |z - 4| \leq 5\sqrt{2}.$$

   Answer: $\mathbf{\frac{25}{4}\sqrt{2}\pi}$

   Solution: Consider the equality case of the above condition. Observe that this constrains to all z such that the sum of the distances between z and $3i$ and z and 4 is constant, which is the definition of an ellipse with foci at those two points. Moreover, since $|z - 3i| + |z - 4| = 5\sqrt{2}$, it isn't hard to see that the major and minor axes of this ellipse are $5\sqrt{2}$ and 5, respectively. To see this, let O be the center of the ellipse. Let P be the point on the ellipse such that OP is perpendicular to the major axes. If the minor axes is 5, then via $45-45-90$ triangles, the sum of the distances from point P to the points $3i$ and 4 is $5\sqrt{2}$. Hence, the minor axis is 5. Therefore, since the area of an ellipse is given by πab, where a and b are half the length of the major and minor axes respectively, the desired area is

$$\frac{5}{2} \cdot \frac{5\sqrt{2}}{2} \cdot \pi = \boxed{\frac{25\sqrt{2}\pi}{4}}.$$

6. Let ABE be a triangle with $AB/3 = BE/4 = EA/5$. Let $D \neq A$ be on line AE such that $AE = ED$ and D is closer to E than to A. Moreover, let C be a point such that $BCDE$ is a parallelogram. Furthermore, let M be on line CD such that AM bisects $\angle BAE$, and let P be the intersection of AM and BE. Compute the ratio of PM to the perimeter of ABE.

   Answer: $\mathbf{\sqrt{5}/8}$

   Solution: The ratio condition on the sides of ABC is equivalent to saying that for some positive real x, $AB = 3x$, $BE = 4x$, and $EA = 5x$, hence ABE is similar to a $3-4-5$ right triangle. Now, consider the homothety centered at A sending E to D. Observe that this same homothety sends P to M, hence $AP : PM = AE : ED = 1 : 1$, thus it suffices to calculate the ratio $AP : AB + BE + EA$. Indeed, from the Angle Bisector Theorem, we have that $BP : PE = AB : AE = 3 : 5$, hence $BP = \frac{3}{2}x$ and $PE = \frac{5}{2}x$. Moreover, since $\angle ABE = 90°$, we can use the Pythagorean Theorem on ABP to find that $AP = \frac{3\sqrt{5}}{2}x$. Therefore, we have the desired ratio is

$$\frac{3/2\sqrt{5}x}{3x + 4x + 5x} = \boxed{\frac{\sqrt{5}}{8}}.$$

7. Points $ABCD$ are vertices of an isosceles trapezoid, with AB parallel to CD, $AB = 1$,$CD = 2$, and $BC = 1$. Point E is chosen uniformly and at random on CD, and let point F be the point on CD such that $EC = FD$. Let G denote the intersection of AE and BF, not necessarily in the trapezoid. What is the probability that $\angle AGB > 30°$?

Answer: $\frac{5-2\sqrt{3}}{2}$

Solution: Let M be the midpoint of CD, Y the midpoint of CM and X the midpoint of DM. Notice that AMD, MBC, and AMB are all equilateral triangles of side length 1. If E is chosen on the segment MC, then the condition is satisfied because as E moves from M to C, the angle $\angle AGB$ increases, and if $E = M$, then $\angle AGB = 60$. If E is chosen off the segment XM, then $\angle BAE \leq 75$, which corresponds to $XE \leq \frac{\sqrt{3}}{2}\tan(15) = \sqrt{3} - \frac{3}{2}$. Finally, if E is chosen in the segment DX, then $\angle AEX \leq 75$, or that $XE \geq \frac{\sqrt{3}}{2}\cot(75) = \sqrt{3} - \frac{3}{2}$. Thus, the total length that E can lie on is $2 - (2\sqrt{3} - 3) = 5 - 2\sqrt{3}$, so the probability is $\boxed{\frac{5-2\sqrt{3}}{2}}$.

8. Let ABC be a triangle with $AB = 13, BC = 14$, and $CA = 15$. Let G denote the centroid of ABC, and let G_A denote the image of G under a reflection across BC, with G_B the image of G under a reflection across AC, and G_C the image of G under a reflection across AB. Let O_G be the circumcenter of $G_AG_BG_C$ and let X be the intersection of AO_G with BC and Y denote the intersections of AG with BC. Compute XY.

Answer: $\frac{196}{197}$

Solution: Perform a homothety with ratio 1/2 about G. Then the image of $G_AG_BG_C$ is the pedal triangle of G with respect to ABC. We claim that the image of O_G lands on the midpoint of G and its isogonal conjugate, the symmedian point K of ABC. Indeed, let G'_A be the image of G_A and define G_B and G_C similarly, and let $K_AK_BK_C$ be the pedal triangle of K with respect to ABC. One can check that

$$AG'_C \cdot AK_C = AG \cdot AK \cdot \cos\angle BAG \cdot \cos\angle BAK = AG'_B \cdot AK_B,$$

hence by the converse of Power of a Point, the points G'_C, G'_B, K_C, and K_B are concyclic. By symmetry, it then follows that $G'_AG'_BG'_C$ and $K_AK_BK_C$ share a circumcircle. Now, since the circumcenter of cyclic hexagon $K_AG'_AK_BG'_BK_CG'_C$ must lie on the perpendicular bisectors of $K_AG'_A$, $K_BG'_B$, and $K_CG'_C$, O'_G clearly is the midpoint of GK, as desired.

Now, if we reverse this homothety to send $G'_AG'_BG'_C$ back to $G_AG_BG_C$, O'_G is sent back to O_G. But since this homothety is centered at G, O'_G is the midpoint of KG, and this homothety has ratio 2, O_G must therefore coincide with K, hence AX is a symmedian of ABC. It is well known that $BX : XC = AB^2 : AC^2$, hence it is a straightforward calculation to get that

$$XY = BY - BX = 7 - \frac{169 \cdot 14}{169 + 225} = \boxed{\frac{196}{197}}.$$

9. Let $ABCD$ be a tetrahedron with $\angle ABC = \angle ABD = \angle CBD = 90°$ and $AB = BC$. Let E, F, G be points on AD, BD, and CD, respectively, such that each of the quadrilaterals $AEFB, BFGC$, and $CGEA$ have an inscribed circle. Let r be the smallest real number such that $\text{area}(EFG)/\text{area}(ABC) \leq r$ for all such configurations A, B, C, D, E, F, G. If r can be expressed as $\frac{\sqrt{a-b\sqrt{c}}}{d}$ where a, b, c, d are positive integers with $gcd(a, b)$ squarefree and c squarefree, find $a + b + c + d$.

Answer: 330

Solution: The key idea here is that the three circles only depend on the length of BD. Because each quadrilateral shares three sides with a triangle, the incircle of the quadrilateral is also the incircle of the triangle. Hence, it follows that for every value of BD, there is exactly one positioning of E, F, G that satisfies the conditions of the problem. This is because there are exactly two planes that are tangent to the three incircles, one of them the base of the triangle. Now, from here we just need to analyze the area of EFG as BD changes. Observe that as BD increases, so does the area of EFG. This is because as D goes to infinity, the tetrahedron approaches a box-like structure, we find that the area of the cross section with the plane will grow as BD approaches infinity. This is because the area of the bigger incircle will grow faster than the area of the smaller incircle, so the tilt of the plane with respect to the base of the tetrahedron will only get larger. Hence if we send BD to infinity and calculate the area of EFG, we can get our least upper bound.

If we send BD to infinity, then each of AD, BD, and CD are perpendicular to the plane of ABC. Naturally, this makes calculation fairly straightforward - fix ABC to have area $1/2$ (i.e. its legs are length 1), and then one can compute that $EG = \sqrt{2}$ easily via symmetry. Now we convert to coordinates. Let $B = (0, 0, 0)$, $A = (0, 1, 0)$, $C = (1, 0, 0)$. Then $E = (0, 1, \sqrt{2})$, $G = (1, 0, \sqrt{2})$ and let $F = (0, 0, k)$. Since $GFBC$ is an inscribed quadrilateral, we have $GF + BC = BF + GC$. By the pythagorean theorem, we have $FG = \sqrt{1 + (\sqrt{2} - k)^2}$, so

$$k + \sqrt{2} = 1 + \sqrt{1 + (\sqrt{2} - k)^2}$$

which yields a solution of $k = \frac{4+\sqrt{2}}{7}$. Then we can compute the height of EFG via the pythagorean theorem which is equal to $\sqrt{\frac{225-96\sqrt{2}}{98}}$. Thus, the ratio is

$$\frac{\frac{\sqrt{2}}{2} \cdot \sqrt{\frac{225-96\sqrt{2}}{98}}}{\frac{1}{2}} = \frac{\sqrt{225 - 96\sqrt{2}}}{7}$$

which yields an answer of $\boxed{330}$.

remark: one can use trigonometry (namely the tangent double angle identity) to find that $\tan \angle FEA = \frac{6\sqrt{2}-4}{7}$ and finish the problem off that way.

10. A $3-4-5$ point of a triangle ABC is a point P such that the ratio $AP : BP : CP$ is equivalent to the ratio $3 : 4 : 5$. If ABC is isosceles with base $BC = 12$ and ABC has exactly one $3-4-5$ point, compute the area of ABC.

Answer: $9\sqrt{2835}$ or $81\sqrt{35}$

Solution: We present a computational solution that relies on two synthetic observations that we present without proof:

Claim 1: Given two points X, Y and a fixed ratio k, the set of all points Z such that $XZ/YZ = k$ is a circle. Furthermore, the line XY passes through the center of the circle. These circles are known as *Apollonius circles*.

Claim 2: Given a triangle $A_1A_2A_3$ and ratios k_1, k_2, k_3, the circles $(A_1, A_2, k_1), (A_2, A_3, k_2)$, and (A_3, A_1, k_3) are coaxial, where (X, Y, k) denotes the locus of all points Z with $XZ/YZ = k$.

Let Γ_A be the set of all points P such that $AP/BP = \frac{3}{4}$, with Γ_B and Γ_C defined similarly. From Claim 1, we know that these are all circles. Now, observe that a $3-4-5$ point must lie on all three of Γ_A, Γ_B, and Γ_C. Since these circles are coaxial, it follows that all three must be tangent to ensure that there is exactly one $3-4-5$ point.

With these synthetic observations, we can now begin computation. Our strategy will take advantage of the following fact: if two circles are externally tangent, then the distance between their centers is the sum of their radii, and if two circles are internally tangent then the distance between their centers is the positive difference of their radii. Let X, Y lie on line BC such that $BX = 8, CX = 10, BY = 72$, and $CY = 90$. Note that since $CX/BX = CY/BY = 5/4$, both these points lie on Γ_B. Moreover, by symmetry, these points are antipodal with respect to Γ_B, hence the segment XY is a diameter and thus the midpoint of XY, which we will denote as M, is the center of Γ_B. Clearly, the radius of Γ_B is $MX = MY = 40$. Now, let the height of ABC be h. Then, from similar reasoning as before, and similar triangles, we can see that the distance from the center of Γ_C and BC is $\frac{16}{7}h$. Moreover, we can use the Pythagorean theorem to get that the radius of Γ_C is $\frac{12}{7}\sqrt{h^2+81}$. Using the Pythagorean theorem again, we get that the distance between the center of Γ_C and the center of Γ_B is $\frac{16}{7}\sqrt{h^2+529}$. We thus have the equation

$$\frac{12}{7}\sqrt{h^2+81} + 40 = \frac{16}{7}\sqrt{h^2+529},$$

which, after some algebra, is equivalent to

$$(h^2+45)(h^2-2835) = 0,$$

hence $h = \sqrt{2835} = 9\sqrt{35}$ and thus the area of ABC is $\boxed{81\sqrt{35}}$.