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1. Bob has 3 different fountain pens and 11 different ink colors. How many ways can he fill his fountain pens with ink if he can only put one ink in each pen?

   Answer: 990

   Solution: He has 11 options to fill his first pen with, 10 for the next pen, and 9 for the last. There are thus $11 \cdot 10 \cdot 9 = \boxed{990}$ ways to fill his pens.

2. At the Berkeley Math Tournament, teams are composed of 6 students, each of whom pick two distinct subject tests out of 5 choices. How many different distributions across subjects are possible for a team?

   Answer: 1694

   Solution: In total, there are 12 tests being taken, and we recognize the only constraint is that no test has more than 6 people taking it. Without the constraint, we have $\binom{16}{4}$ ways to divide the 12 tests among 5 subjects. We now need to remove all the partitions with more than 6 in a given subject. There are 5 ways to choose which subject gets at least 7 tests, and we can freely place the remaining 5 tests. As such, there are $5 \cdot \binom{9}{4}$ invalid arrangements, so we our answer is $\binom{16}{4} - 5 \cdot \binom{9}{4} = 1820 - 126 = \boxed{1694}$.

3. Consider the 9×9 grid of lattice points $\{(x, y) \mid 0 \leq x, y \leq 8\}$. How many rectangles with nonzero area and sides parallel to the x, y axes are there such that each corner is one of the lattice points and the point $(4, 4)$ is not contained within the interior of the rectangle? ((4,4) is allowed to lie on the boundary of the rectangle).

   Answer: 1040

   Solution: We note that the bottom left and top right corner of a square uniquely determine it. Ignoring the constraint that $(4, 4)$ is not within the square, we then have $\binom{9}{2}$ ways to choose the x coordinates of the the corners, and $\binom{9}{2}$ ways to choose the y coordinates, giving 36^2 squares in total.

   We note that squares contain $(4, 4)$ iff the bottom left corner is within the bottom left quadrant of the grid and the top right corner is in the top right quadrant (where neither quadrant includes the points along the lines $x = 4$, $y = 4$). There are thus 4^2 ways to choose corners in each quadrant, and thus 16^2 squares that include $(4, 4)$.

   Our answer is thus $36^2 - 16^2 = (36 - 16)(36 + 16) = 20 \cdot 52 = \boxed{1040}$.

4. Alice starts with an empty string and randomly appends one of the digits $2, 0, 1$, or 8 until the string ends with 2018. What is the probability Alice appends less than 9 digits before stopping?

   Answer: $\frac{1279}{2^{16}}$

   Solution: The probability the last 4 digits of any sequence is 2018 is $\frac{1}{4^4} = \frac{1}{256}$. The probability we finish after appending n digits is the probability the last 4 are 2018, and we did not end before n digits. However, if our last digits are 2018, that implies we did not end with 2018 after $n - 1, n - 2$ or $n - 3$ digits. Therefore, the probability that we end at the 4th, 5th, 6th, or 7th digits is $\frac{1}{256}$ each. The probability of ending at the 8th digit is then $\frac{1}{256} - \frac{1}{256^2}$, since we need avoid overcounting the case of 20182018. Our answer is thus

$$4 \cdot \frac{1}{256} + \frac{255}{256^2} = \frac{5 \cdot 256 - 1}{256^2} = \boxed{\frac{1279}{2^{16}}}$$

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5. Alice and Bob play a game where they start from a complete graph with n vertices and take turns removing a single edge from the graph, with Alice taking the first turn. The first player to disconnect the graph loses. Compute the sum of all n between 2 and 100 inclusive such that Alice has a winning strategy. (A complete graph is one where there is an edge between every pair of vertices.)

   Answer: 2575

   Solution: If the graph is currently a tree, then the next move must disconnect the graph. Otherwise, there is a move that does not disconnect the graph. Each complete graph of n vertices has $\binom{n}{2}$ edges, and any tree with n vertices has $n-1$ edges, so when players play optimally (note that if the graph is not a tree, there exists a cycle, so there exists a edge which a player can remove), $\binom{n}{2}-(n-1)+1$ are removed at the end of the game. Which player wins then depends only on the parity of $\binom{n}{2}-n$, where player A wins if it is even and B wins if it is odd.

   We note that $\binom{n}{2}=\frac{n(n-1)}{2}$ is even iff $n \equiv 0 \mod 4$ or $n \equiv 1 \mod 4$, so $\binom{n}{2}-n$ is even iff $n \equiv 0 \mod 4$ or $n \equiv 3 \mod 4$. The sum of all valid n is then $4 \cdot \frac{25 \cdot 26}{2} + 3 \cdot 25 + 4 \cdot \frac{24 \cdot 25}{2} = \boxed{2575}$

6. Compute
$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{i+j}{i}3^{-(i+j)}.$$

   Answer: 3

   Solution: We note that we can rearrange the sum as $\sum_{i=0}^{\infty} 3^{-i}\sum_{j=0}^{i}\binom{i}{j}$. Recall that $\sum_{j=0}^{i}\binom{i}{j} = 2^i$. We thus see the sum is $\sum_{i=0}^{\infty}(2/3)^i = \boxed{3}$.

7. Let S be the set of line segments between any two vertices of a regular 21-gon. If we select two distinct line segments from S at random, what is the probability they intersect? Note that line segments are considered to intersect if they share a common vertex.

   Answer: $\frac{5}{11}$

   Solution: Consider the case where the diagonals we choose do not share vertices. In this case, the probability they intersect is simply $1/3$, since if we select any 4 points, there are 3 ways to choose which pairs get connected, and only one of which has the diagonals intersecting. There are $n \cdot \binom{n-1}{2}$ ways to select pairs of segments that share a vertex, $\binom{n}{2}\binom{n-2}{2}/2$ ways to select pairs of segments that don't share a vertex, and $\binom{\binom{n}{2}}{2}$ ways to select distinct pairs of segments. The

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probability we select non-intersecting line segments is thus

$$\begin{aligned}\frac{\frac{\binom{n}{2}\binom{n-2}{2}}{6}+n\binom{n-1}{2}}{\binom{\binom{n}{2}}{2}} &= \frac{\frac{\binom{n}{2}\binom{n-2}{2}}{6}+\frac{n(n-1)(n-2)}{2}}{\frac{\binom{n}{2}\left(\binom{n}{2}-1\right)}{2}} \\ &= \frac{\frac{\binom{n-2}{2}}{3}+2(n-2)}{\binom{n}{2}-1} \\ &= \frac{\frac{(n-2)(n-3)}{6}+2(n-2)}{\binom{n}{2}-1} \\ &= \frac{(n-2)(n+9)}{6\left(\binom{n}{2}-1\right)} \\ &= \frac{19\cdot 30}{6\,(210-1)} \\ &= \frac{19\cdot 5}{209} \\ &= \boxed{\frac{5}{11}}\end{aligned}$$

An alternative way of computing it would be noting that the total number of distinct pairs of segments is equal to the sum of the number of pairs with no shared vertices and the number of pairs with a shared vertex, so we have

$$\begin{aligned}\frac{1}{3}+\frac{2}{3}\frac{n\binom{n-1}{2}}{\frac{\binom{n}{2}\left(\binom{n}{2}-1\right)}{2}} &= \frac{1}{3}+\frac{\frac{4}{3}(n-2)}{\binom{n}{2}-1} \\ &= \frac{1}{3}+\frac{4/3}{11} \\ &= \frac{15}{33} \\ &= \boxed{\frac{5}{11}}.\end{aligned}$$

8. Moor and nine friends are seated around a circular table. Moor starts out holding a bottle, and whoever holds the bottle passes it to the person on his left or right with equal probability until everyone has held the bottle. Compute the expected distance between Moor and the last person to receive the bottle, where distance is the fewest number of times the bottle needs to be passed in order to go back to Moor.

Answer: $\mathbf{\frac{25}{9}}$

Solution: Consider some person X that is not Moor, and let L and R be the people to X's left and right respectively. For X to be the last person, the bottle must have come to L first, then gone all the way around to R before touching X, or have gone to R first then come around to L before reaching X. We note regardless of whether L or R was visited first, the probability of going around before reaching X is the same. We thus conclude that since at least one of L or R must have been visited before X and it does not matter whether L or R was visited first, each

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of Moor's friends is equally likely to be the last to touch the bottle. The expected distance is then $\frac{1+2+3+4+5+4+3+2+1}{9} = \boxed{\frac{25}{9}}$.

9. Let S be the set of integers from 1 to 13 inclusive. A permutation of S is a function $f : S \to S$ such that $f(x) \neq f(y)$ if $x \neq y$. For how many distinct permutations f does there exists an n such that $f^n(i) = 13 - i + 1$ for all i.

Answer: 7065

Solution: Split into cycle structures, and note that i and $13 - i + 1$ must lie in the same cycle, at opposite ends, and that 7 is in a cycle on its own. We thus need to divide 6 pairs into groups and arrange each group in an appropriate cycle. We note that the cycles with k pairs are in the correct position for all $n \equiv k \mod 2k$, so to make sure there is an n such that all cycles are in the correct phase at the same time, so the only possible ways to partition the 6 pairs into groups are $(1, 1, 1, 1, 1, 1), (1, 1, 1, 3), (1, 5), (3, 3), (2, 2, 2), (6)$.

In the case where the pairs are partitioned as $(1, 1, 1, 1, 1, 1)$, there is only one way to arrange the cycles.

In the $(1, 1, 1, 3)$ case, there are $\binom{6}{3}$ ways to select which pairs are in the cycle of 3. Since the cycle of 3 is rotationally invariant, we arbitrarily fix the locations of one pair, and there are $2! \cdot 2^2$ ways to arrange the remaining two pairs. There are thus $20 \cdot 8 = 160$ permutations in this case.

In the $(1, 5)$ case, we have 6 ways to select which pairs are in the cycle of 5, and $4! \cdot 2^4$ ways to arrange the cycle of 5. In total, we have $6 \cdot 24 \cdot 16 = 2304$ ways.

$(3, 3)$ case: There are $\binom{6}{3}/2$ ways to split the pairs into two cycles, and in each cycle, $2! \cdot 2^2$ to arrange them. We thus have $10 \cdot 8 \cdot 8 = 640$ ways.

$(2, 2, 2)$ case: There are 15 ways to split the pairs into 3 groups of 2. In each pair of 2, there are 2 ways to arrange the cycle, so there are $15 \cdot 2^3 = 120$ ways.

In the (6) case, there are $5! \cdot 2^5$ ways of arranging it, leading to $120 * 32 = 3840$ permutations.

In total, we have $1 + 160 + 640 + 2304 + 120 + 3840 = \boxed{7065}$ permutations.

10. Consider a $2 \times n$ grid where each cell is either black or white, which we attempt to tile with 2×1 black or white tiles such that tiles have to match the colors of the cells they cover. We first randomly select a random positive integer N where N takes the value n with probability 2^{-n}. We then take a $2 \times N$ grid and randomly color each cell black or white independently with equal probability. Compute the probability the resulting grid has a valid tiling.

Answer: $\mathbf{\frac{9}{23}}$

Solution: Let p_n be the probability that a random $2 \times n$ grid has a valid tiling. Consider the cell at an end of the first row of the grid. If the cell below it is the same color and and there exists a valid tiling, then there exists a valid tiling where we cover those two cells with a vertical tile of the same color, and the remaining $2 \times n - 1$ grid is tiled in a valid manner. If the cell below is not the same color, then if there was a valid tiling, then there must be be a pair of horizontal tiles of different colors to cover the first two columns, with a valid tiling over the remaining $2 \times n - 2$ grid. We thus see that p_n satisfies the recurrence $p_n = \frac{1}{2}p_{n-1} + \frac{1}{8}p_{n-2}$, with $p_0 = 1, p_1 = \frac{1}{2}$.

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We see the generating function $P(x) = \sum_{n=0}^{\infty} p_n x^n$ satisfies

$$\begin{aligned} P(x) &= 1 + \frac{1}{2}x + \sum_{n=2}^{\infty} p_n x^n \\ &= 1 + \frac{1}{2}x + \sum_{n=2}^{\infty} \left(\frac{1}{2}p_{n-1} + \frac{1}{8}p_{n-2} \right) x^n \\ &= 1 + \frac{1}{2}x + \frac{x}{2}\sum_{n=1}^{\infty} p_n x^n + \frac{x^2}{8}\sum_{n=0}^{\infty} p_n x^n \\ &= 1 + \frac{1}{2}x + \frac{x}{2}\left(P(x) - 1\right) + \frac{x^2}{8}P(x) \\ P(x) &= \frac{1}{1 - \frac{x}{2} - \frac{x^2}{8}}. \end{aligned}$$

We then note that the probability we want is

$$\sum_{n=1}^{\infty} \frac{p_n}{2^n},$$

which is simply $P(x) - 1$ evaluated at $x = \frac{1}{2}$, so is equal to $\frac{1}{1 - \frac{1}{4} - \frac{1}{32}} - 1 = \frac{32}{23} - 1 = \boxed{\frac{9}{23}}$.