1. What is the integer part of the following expression, which contains 2018 square roots?

$$\sqrt{2018+\sqrt{2018+\sqrt{2018+\dots}}}$$

Answer: 45

Solution: Let

$$y=\sqrt{2018+\sqrt{2018+\sqrt{2018+\dots}}}$$

Then $y^2 = 2018 + y$, so solving for y, we have

$$y=\frac{1+\sqrt{8073}}{2}$$

We see that $90^2 = 8100$, so the integer part is

$$\frac{1+89}{2}=\boxed{45}$$

(Note that $\sqrt{2018+\sqrt{2018}}$ has integer part 45).

2. Let $a_{n+1} = \frac{a_n+b_n}{2}$ and $b_{n+1} = \frac{1}{\frac{1}{a_n}+\frac{1}{b_n}}$, with $a_0 = 13$ and $b_0 = 29$. What is $\lim_{n\to\infty} a_nb_n$?

Answer: 0

Solution: Let us look at a_nb_n. $a_nb_n = \frac{a_n+b_n}{2\frac{a_n+b_n}{a_nb_n}} = \frac{a_n+b_n}{\frac{2}{a_n}+\frac{2}{b_n}} = \frac{1}{2}a_{n+1}b_{n+1}$ Thus, a_nb_n decreases. Thus, the limit is just $\boxed{0}$

3. What is the 100th derivative of $f(x) = e^x \cos(x)$ at $x = \pi$?

Answer: $4^{25}e^{\pi}$

Solution: We begin by looking at the first few derivatives: $f'(x) = e^x(\cos(x) - \sin(x))$ $f''(x) = -2e^x \sin(x)$ $f'''(x) = -2e^x(\cos(x) - \sin(x))$ $f^{(4)}(x) = -4e^x \cos(x) = -4f(x)$ Continuing this on to the 100th derivative: $f^{(100)} = (-4)^{25}f(x) = -4^{25}f(x)$ So at $x = \pi$, the derivative is just $\boxed{4^{25}e^{\pi}}$

4. Compute the following limit:

$$\lim_{n\to\infty}\int_0^1 \frac{nx^n}{\sqrt{4x^3-x+1}}dx$$

Answer: $\frac{1}{2}$

Solution: We use integration by parts. Let $u = \frac{1}{\sqrt{4x^3-x+1}}$ and $dv = nx^n$. Then

$$du=-\frac{1}{2}\frac{12x^2-1}{\sqrt{(4x^3-x+1)^3}}$$

$$v=\frac{nx^{n+1}}{n+1}$$

Thus, we have

$$\int_0^1 \frac{nx^n}{\sqrt{4x^3-x+1}}dx + \int_0^1 \left(\frac{nx^{n+1}}{n+1}\right)\left(-\frac{1}{2}\frac{12x^2-1}{\sqrt{(4x^3-x+1)^3}}\right)dx = \frac{1}{2}\frac{n}{n+1}$$

Note that

$$\frac{n}{n+1} \to 1$$

as n gets large and $x^{n+1} \to 0$ as n gets large for $x \in [0,1)$. Hence, the second integral goes to 0 as n gets large, and the right hand side goes to $\frac{1}{2}$. Therefore,

$$\int_0^1 \frac{nx^n}{\sqrt{4x^3-x+1}} = \frac{1}{2}$$

as desired.

5. What is

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos x}{1+e^{-x}}dx$$

Answer: -1

Solution: Let f be an even function. Consider the following integral:

$$\int_{-a}^{a} \frac{f(x)}{1+e^{-x}}dx$$

By symmetry:

$$I = \int_{-a}^{a} \frac{f(x)}{1+e^{-x}}dx = \int_{-a}^{a} \frac{f(x)}{1+e^{x}}dx \implies 2I = \int_{-a}^{a} \frac{f(x)}{1+e^{-x}} + \frac{f(x)}{1+e^{x}}dx = \int_{-a}^{a} f(x)dx$$

Now letting $f(x) = \cos(x)$, we obtain that

$$2I = \int_{-\pi/2}^{\pi/2} \cos(x)dx = 2$$

so the answer is $\boxed{1}$.

6. What is the value of:

$$\sum_{n=1}^{\infty} \prod_{k=1}^{2n} \cos\frac{k\pi}{2n+1}$$

Answer: $-\frac{1}{3}$

Solution: We begin by realizing that $\prod_{k=1}^{2n} \cos\frac{k\pi}{2n+1}$ is $(-1)^n(4)^{-n}$ Thus, the sum becomes $\sum_{n=1}\infty(-1)^n(4)^{-n}$, which is simply $\boxed{-\frac{1}{3}}$

7. What is the following limit:

$$\lim_{x\to 0} \frac{\tan(3x)\sin(4x) + \sin(5x)\tan(2x)}{\tan(6x)\sin(7x)\cos(8x)}$$

Answer: $\mathbf{\frac{11}{21}}$

Solution: To evaluate this limit, begin by replacing $\tan(nx)$ with nx, $\sin(nx)$ with nx, and $\cos(nx)$ with 1, as these are the rough approximations of sin, cos, and tan near 0. Plugging these into the original limit: $\lim_{x\to 0} \frac{\tan(3x)\sin(4x)+\sin(5x)\tan(2x)}{\tan(6x)\sin(7x)\cos(8x)} = \lim_{x\to 0} \frac{(3x)(4x)+(5x)(2x)}{(6x)(7x)}$ This limit can then be simplified to $\boxed{\frac{11}{21}}$

8. What is the maximum radius of a circle tangent to the curves $y = e^{-x^2}$ and $y = -e^{-x^2}$ at two points each?

Answer: $\mathbf{\sqrt{\frac{1}{2}(\ln(2)+1)}}$

Solution: Let us denote the radius as r, with x and y components x_r and y_r. From this, we have $x_r^2 + y_r^2 = r^2$, and the slope of the radial vector is $m = \frac{y}{x}$. This must be perpendicular to the curves, so we can solve: $\frac{-x}{y} = -2xe^{-x^2}$ Solving this equation and plugging in our values of x_r and y_r into r, we arrive that $r = \sqrt{\frac{1}{2}(\ln(2)+1)}$

9. Compute

$$\int_{-\infty}^{0} \frac{1}{x^3 - 1} dx$$

Answer: $\mathbf{-\frac{2\pi}{3\sqrt{3}}}$

Solution: We first perform partial fractions on

$$\frac{1}{x^3-1}$$

to obtain that

$$\int \frac{1}{x^3-1} dx = \int \frac{-x-2}{3(x^2+x+1)} + \frac{1}{3(x-1)} dx$$

$$\int \frac{1}{3(x-1)} dx = \frac{1}{3}\ln(x-1)$$

$$\int \frac{-x-2}{x^2+x+1} dx = \frac{1}{3}\int -\frac{2x+1}{2(x^2+x+1)} - \frac{3}{2(x^2+x+1)} dx$$

$$\frac{1}{3}\int -\frac{2x+1}{2(x^2+x+1)} - \frac{3}{2(x^2+x+1)} dx = -\frac{1}{6}\int \frac{2x+1}{x^2+x+1} - \frac{1}{2}\int \frac{1}{x^2+x+1} dx$$

Perform the substitution: $u = x^2 + x + 1$

$$-\frac{1}{6}\int \frac{2x+1}{x^2+x+1} dx = -\frac{1}{6}\int \frac{1}{u} du = -\frac{\ln(u)}{6}$$

$$-\frac{1}{2}\int \frac{1}{x^2+x+1} dx = -\frac{1}{2}\int \frac{1}{(x+\frac{1}{2})^2+\frac{3}{4}} dx$$

$$-\frac{1}{2}\int \frac{1}{(x+\frac{1}{2})^2+\frac{3}{4}}dx = -\frac{1}{\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}}$$

$$\int \frac{1}{x^3-1} = \frac{1}{3}\ln(1-x) - \frac{1}{6}\ln(x^2+x+1) - \frac{1}{\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}}$$

Evaluating this at $x = -\infty$ and $x = 0$, we conclude that:

$$\int_{-\infty}^{0} \frac{1}{x^3-1} = \boxed{-\frac{2\pi}{3\sqrt{3}}}$$

10. Let T be defined by the recurrence relation $T_{n+1} = 2xT_n - T_{n-1}$ with $T_0 = 1$ and $T_1 = x$. What is

$$\sum_{n=2}^{\infty}\int_0^1 T_n dx$$

Answer: -1

Solution: First, we claim that

$$\int_0^1 T_n dx = \frac{n\sin\frac{n\pi}{2} - 1}{n^2-1}$$

To prove this, note that T_n is the nth Tchebyshev polynomial. So in fact,

$$\int_0^1 T_n dx = -\int_{\frac{\pi}{2}}^{0} \cos(nx)\sin(x)dx = \int_0^{\frac{\pi}{2}} \cos(nx)\sin(x)dx$$

and using integration by parts, we have

$$\int \cos(nx)\sin(x)dx = \frac{n\sin(x)\sin(nx) + \cos(x)\cos(nx)}{n^2-1} + C$$

so plugging in the limits we get our desired result. Thus,

$$\sum_{n=2}^{\infty}\int_0^1 T_n dx = \sum_{n=2}^{\infty}\frac{n\sin\frac{n\pi}{2} - 1}{n^2-1} = \sum_{n=1}^{\infty} -\frac{1}{4n^2-1} + \frac{1}{4n+2} - \frac{1}{4n-2}$$

which "morally" converges to $\boxed{-1}$ by telescoping. (Note: a full rigorous proof will need background from real analysis, so will be omitted here).