Time limit: 60 minutes.
Maximum score: 80 points.
Instructions: For this test, you work in teams of six to solve a multi-part, proof-oriented question.

Problems that use the words "compute", "list", or "draw" only call for an answer; no explanation or proof is needed. Unless otherwise stated, all other questions require explanation or proof. Answers should be written on sheets of scratch paper, clearly labeled, with every problem *on its own sheet*. If you have multiple pages for a problem, number them and write the total number of pages for the problem (e.g. 1/2, 2/2).

Write your team ID number clearly on each sheet. Only submit one set of solutions for the team. Do not turn in any scratch work. After the test, put the sheets you want graded into your packet. If you do not have your packet, ensure your sheets are labeled *extremely clearly* and stack the loose sheets neatly.

In your solution for a given problem, you may cite the statements of earlier problems (but not later ones) without additional justification, even if you haven't solved them.

The problems are ordered by content, NOT DIFFICULTY. It is to your advantage to attempt problems from throughout the test.
No calculators.

1 Introduction

Tropical geometry is a relatively new area in mathematics. Loosely described, it is a piece-wise linear version of algebraic geometry, over a particular structure known as the *tropical semiring*. Tropical algebraic geometry establishes and studies some general principles to translate problems in algebraic geometry into purely combinatorial ones. This power round will give a brief introduction of some of the basic concepts used in tropical geometry.

2 Tropical Arithmetic

We begin by defining how to do arithmetic. In tropical arithmetic, we use the operators $\oplus$ and $\odot$ in lieu of the usual $+$ and $\times$ in classical arithmetic. These operators are defined on $\mathbb{R} \cup \infty$; that is, the set of real numbers, plus one special "number", ∞, as follows:

$$x \oplus y = \min\{x, y\}$$

$$x \odot y = x + y.$$

For example, the tropical sum $4 \oplus 9 = 4$, and the tropical product $4 \odot 9 = 13$.

1. Compute the following:

   (a) **[1]** $3 \oplus 14$,

   (b) **[1]** $3 \odot 14$,

   (c) **[1]** $2 \oplus \infty$,

   (d) **[1]** $0 \oplus 1$.

For this power round, we will be working on the set $\mathbb{R} \cup \infty$ equipped with these two operations $\oplus$ and $\odot$. This structure enjoys many of the properties of ordinary arithmetic. For example, these two operations are commutative:

$$x \oplus y = y \oplus x, \text{ and } x \odot y = y \odot x,$$

and associative:

$$x \oplus (y \oplus z) = (x \oplus y) \oplus z, \text{ and } x \odot (y \odot z) = (x \odot y) \odot z.$$

Order of operations is as usual – multiplication comes before addition, just as you learned in elementary school.

2. **[3]** Prove the distributive law:

$$x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z).$$

The multiplicative identity on this structure is 0; that is to say, $x \odot 0 = x$ for any number x.

3. **[3]** Find the additive identity, which is an element of this set y such that $x \oplus y = x$ for all x.

One feature present in ordinary arithmetic that is missing from tropical arithmetic, however, is subtraction. We cannot define $9 \ominus 4$ meaningfully because $4 \oplus x = 9$ has no solution. However, tropical division *can* be defined, as ordinary subtraction. Satisfying the properties above (commutativity of addition, associativity, additive and multiplicative identities, distributivity, and additive identity being the zero multiplicative element) makes this structure a *semiring*, henceforth called the *tropical semiring*.

3 Tropical Polynomials

We define exponentiation in the usual manner as repeated multiplication, writing, for example, $x^3 = x \odot x \odot x$. In this way, we can have tropical monomials, such as $x_1^2 x_2^{-1} x_3 = x_1 \odot x_1 \oslash x_2 \odot x_3 = x_1 + x_1 - x_2 + x_3$, which are simply linear functions. We note that conversely, any linear function with integer coefficients can be expressed as a tropical monomial.

Tropical polynomials, as you might expect, are then finite tropical sums, or linear combinations, of tropical monomials, and every tropical polynomial in n variables is a function $\mathbb{R}^n \to \mathbb{R}$. Because tropical addition and multiplication are commutative, associative, and distributive, we can multiply tropical polynomials just as we can with ordinary polynomials. For example,

$$(x \oplus 1) \odot (x \oplus -1) = x \odot x \oplus x \odot 1 \oplus x \odot -1 \oplus 1 \odot -1 = x^2 \oplus x \odot -1 \oplus 0.$$

Note that the last equivalence holds because $x \odot -1$ is always less than $x \odot 1$, so the latter term can be removed.

4. **[2]** Compute $(x \oplus 2 \odot y)^2$.

5. **[4]** Show that the *freshman's dream* holds in tropical arithmetic:

$$(x \oplus y)^n = x^n \oplus y^n$$

for all x, y in the tropical semiring and all integers n.

When evaluating a tropical polynomial in classical arithmetic, what we obtain is the minimum of a finite collection of linear functions.

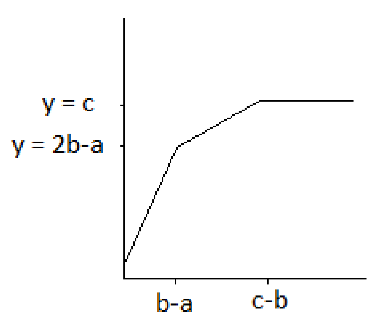
6. **[6]** Show that such functions $p : \mathbb{R} \to \mathbb{R}$ are *concave*; that is, they satisfy the property that

$$p\left(\frac{x_1 + x_2}{2}\right) \geq \frac{p(x_1) + p(x_2)}{2}$$

for all x_1 and x_2. (Note that addition and division in this problem are classical.)

4 Graphs and Factors

We begin by examining tropical functions of one variable. For example, consider the second-degree tropical polynomial $y = a \odot x^2 \oplus b \odot x \oplus c$. The graph of this function consists of three lines: $y = a + 2x$, $y = b + x$, and $y = c$, and the value of $p(x)$ at any point is given by the least of the three at that point.



We note that the three lines $y = a + 2x$, $y = b + x$, and $y = c$ intersect at the points $x = b - a$ and $x = c - b$. If $b - a \le c - b$, then all three lines actually contribute to the graph of our function. We can then break down the polynomial as a tropical product of linear factors by noting that $p(x) = a \odot (x \oplus (b - a)) \odot (x \oplus (c - b))$, and call the points $x = b - a$ and $x = c - b$ the roots of our quadratic equation.

We claim that this is true in general: every single-variable tropical polynomial function can be written uniquely as a tropical product of tropical linear functions. This statement is the *Tropical Fundamental Theorem of Algebra*.

Note that this uniqueness applies to *functions*; distinct tropical polynomials can represent the same function. For example,

$$x^2 \oplus 17 \odot x \oplus 2 = x^2 \oplus 1 \odot x \oplus 2 = (x \oplus 1)^2.$$

The Tropical Fundamental Theorem states that these tropical polynomials can be replaced by an equivalent one that is the tropical product of linear factors.

7. Compute the factorization for the tropical polynomials
   (a) **[2]** $x^2 \oplus 6 \odot x \oplus 28$
   (b) **[2]** $x^2 \oplus 14$
   (c) **[2]** $x^2 \oplus 4 \odot x \oplus 8$
   (d) **[2]** $x^3 \oplus 3 \odot x^2 \oplus 3 \odot x \oplus 1$.

8. (a) **[6]** Find a general rule for factoring the polynomial $a \odot x^2 \oplus b \odot x \oplus c$. Prove that your rule works.
   (b) **[3]** How is this analogous to the quadratic formula in classical arithmetic?

9. **[10]** Show that every single-variable tropical polynomial can be factored uniquely into a tropical product of tropical linear functions.

10. **[3]** Give an example of a tropical polynomial p of two or more variables, and two different factorizations of p into irreducibles, or functions that cannot be factored further. Thus, we note that the Fundamental Theorem of Algebra does not hold for tropical multivariable polynomials.

5 Tropical Eigenvalues

Matrix multiplication generalizes tropically as well. We define the product of two tropical matrices A and B similarly to classical matrix multiplication: the entry in the ith row and jth column of AB is the number $a_{i1} \odot b_{1j} \oplus ... \oplus a_{in} \odot b_{nj}$, where a_{ij} and b_{ij} denote the entries of A and B, respectively, on the ith row and jth column. For example,

$$\begin{pmatrix} 3 & 3 \\ 0 & 7 \end{pmatrix} \odot \begin{pmatrix} 4 & 1 \\ 5 & 2 \end{pmatrix} = \begin{pmatrix} 7 & 4 \\ 4 & 1 \end{pmatrix}.$$

11. **[2]** Compute the tropical product

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \odot \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

Let A be an $n \times n$ matrix. An *eigenvalue* of A is a real number λ for which $A \cdot v = \lambda v$ for some $v \in \mathbb{R}^n$; the vector v is then called an *eigenvector* of A. This has a tropical equivalent, as well: an eigenvalue of a tropical matrix A is a number λ for which $A \odot v = \lambda \odot v$ for some vector v.

12. Compute tropical eigenvalues of the matrices

   (a) **[4]**

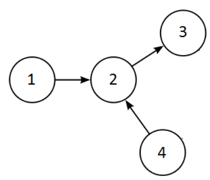
$$\begin{pmatrix} \infty & 1 \\ 1 & \infty \end{pmatrix}$$

   (b) **[4]**

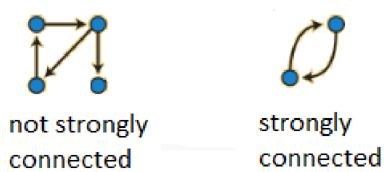
$$\begin{pmatrix} \infty & 1 & 0 \\ 0 & \infty & 1 \\ 1 & 0 & \infty \end{pmatrix}.$$

13. Let G be a directed graph, and label the vertices of G as $1, \dots, n$. An adjacency matrix of a graph G is a matrix A where a_{ij} is the weight of the edge from vertex i to vertex j, if such an edge exists. If it does not exist, we consider the weight of that edge to be ∞.

   (a) **[2]** Find the adjacency matrix of the following graph. Assume all edges shown have weight 1.



   (b) **[2]** A directed graph is called *strongly connected* if there is an (oriented) path in each direction between each pair of vertices of the graph.



   Is the graph in part (a) strongly connected? Why or why not?

   (c) **[10]** If A is the adjacency matrix of a strongly connected directed graph, what can we say about its eigenvalues?

6 Bonus: We Bet You Want To Know What's On The T-Shirt

We have defined a tropical polynomial function $p : \mathbb{R}^n \to \mathbb{R}$ as the minimum of a finite set of linear functions. For each such p, we now define the *hypersurface* $V(p)$ to be the set of all points in $\mathbb{R}^n$ for which this minimum is attained at least twice.

For instance, if $p(x) = a \odot x^2 \oplus b \odot x \oplus c$ and $b - a < c - b$, then the hypersurface consists of the two points $x = b - a$ and $x = c - b$, the roots of the polynomial.

14. **[4]** Graph the hypersurface for the polynomial in two variables

$$p(x, y) = -1 \odot x \odot y \oplus x \oplus y \oplus 0.$$

7 Conclusion

Tropical geometry has many applications. It lies at the interface between algebraic geometry and combinatorics, with connections to many other areas. It has uses in linear optimization and dynamic programming, such as in job scheduling, location analysis, or finding shortest paths of a weighted directed graph. It is used to solve integer programs, for statistical inference, and in at least one case, to design auctions. Classical objects (lines, polynomials, and curves) can be transformed into tropical objects while preserving many of their characteristics, as well as vice versa, providing a new way of analyzing structures in algebraic geometry and simplifying the construction of curves with desired properties.

8 Trivia

Tropical geometry was initially known as max-plus geometry (from a related strain that takes the convention of $x \oplus y = \max(x, y)$). French mathematicians settled on the term "tropical geometry" to honor their colleague Imre Simon, who worked on the subject in Brazil.