

Time Limit: 60 mins.

Maximum Score: 100 points.

Instructions:

1. All problems require justification unless stated otherwise.
2. You may freely assume results of a previous problem in proving later problems, even if you have not proved the previous result.
3. You may use both sides of the paper and multiple sheets of paper for a problem, but separate problems should be on separate sheets of paper. Label the pages of each problem as 1/2, 2/2, etc., in the upper right hand corner. Write your team ID at the upper-right corner of every page you turn in.
4. Partial credit may be given for partial progress on a problem, provided the progress is sufficiently nontrivial.
5. **Calculators are not allowed!**

Introduction: This power round deals with projective planes, which are geometries without parallel lines. These structures have applications in multiple branches of mathematics: Finite Projective Planes are useful in Algebra and Combinatorics, and the Real and Complex Projective Planes are useful in Geometry and Topology. In this round, we will consider special examples of projective planes and also examine just a few of their properties.

Introduction to Projective Planes (35 Points)

A **projective plane** consists of a set of points P , a set of lines L , and an incidence relation $I \subseteq P \times L$. If $(p, l) \in I$, we say that the point p is incident to the line l (or vice-versa). The incidence relation is required to satisfy the following properties:

- Given any two distinct points, there is a unique line incident to both of them. I.e., there is a unique line through any two points.
- Given any two distinct lines, there is a unique point incident to both of them. I.e., any two lines intersect at a unique point.
- (Non-degeneracy) There are four points, no three of which are incident to the same line.

From now on, for the sake of brevity, we will often use the notation p_1p_2 to denote the unique line incident to the points p_1 and p_2 , and we will use the notation $l \cap l'$ to denote the unique point incident to the lines l and l' . However, remember that these “points” and “lines” are just elements of some set, and they don’t necessarily have to correspond to what we usually think of as points and lines. We will now prove some basic properties about projective planes.

- For our first example, the following image shows a projective plane with 7 points and lines (the circle is also a line, as are the six line segments). This projective plane is known as the Fano Plane. For the following questions about the Fano Plane, no proof is required.



- [2] For the Fano Plane, how many elements does I , the set encoding the incidence relation, contain? In other words, how many pairs (p, l) , consisting of a point and a line in the Fano Plane, are there with p incident to l ?

Solution: There are 7 lines and 3 points on each line, giving 21 incident pairs.

- [2] Redraw the Fano Plane and circle four points, no three of which are collinear. This will show that the Fano Plane satisfies the non-degeneracy condition.

Solution: One choice is the 3 points on the circle, in addition to the center point.

- [2] For the previous part, how many such quadruples of points are there in the Fano Plane?

Solution: There are $\binom{7}{4} = 35$ choices of 4 points. If four points fail to satisfy condition 3, then since every line has 3 points, 3 of the points must be on the same line, and the fourth point can be any other point. To choose these points is thus equivalent to choosing a line (and thus all three of its points) and a point not on the line. There are 7 lines and 4 points not on any fixed line, giving 28 choices that fail the 3rd condition. Thus, the number of choices that satisfy the condition is $35 - 28 = 7$.

- [2] Compute the number of permutations f of the 7 points of the Fano Plane such that if p_1, p_2, p_3 are on the same line, so are $f(p_1), f(p_2), f(p_3)$. (Such a permutation is also called a collineation, and collineations will be studied in the final section.)

Solution: If we choose the images of the central point and the bottom left and bottom right points, the collineation is determined, since the three points on the circle must be sent to the 3rd point on each of the 3 lines spanned by the images of the first three points, and then the 7th point, at the top, has one choice of image.

We can send the bottom left point to any of the 7 points, and then send the bottom right point to any of the 6 remaining points. The center point can then be sent to any point not on the same line as the images of the first two points, giving 4 choices. Thus, the total number of collineations of the Fano Plane is $7 \cdot 6 \cdot 4 = 168$.

- Now, we will use the non-degeneracy condition to prove some general facts about projective planes.

- a. [2] Prove that a projective plane must have at least 6 lines.

Solution: Let a, b, c, d be four points, no three of which are collinear. Then, we claim that the lines ab, ac, ad, bc, bd, cd are six distinct lines. If two of them were in fact the same line, it would contain either 3 or 4 of the points, contradicting no three of a, b, c, d being collinear. Thus, any projective plane has at least 6 lines.

- b. [2] Prove that a projective plane must have at least 5 points.

Solution: As in the previous part, let a, b, c, d be points such that no three are collinear. Since ab and cd are distinct lines (as shown above), they must intersect at some point p . If p were a or b , then line cd would also contain that point in addition to cd ; if p were c or d , then line ab would contain that point in addition to a and b . In either case, three of the points a, b, c, d would be on the same line, a contradiction. Thus, p must in fact be a new, fifth point. Thus, every projective plane has at least 5 points.

- c. [2] Show that any projective plane contains four lines, no three of which are incident to the same point.

Solution: Let a, b, c, d be four points, no three of which are collinear. Then, the lines ab, ac, ad, bc, bd, cd are six distinct lines. Consider the four lines ab, bc, cd, ad . Assume for contradiction that three of them are concurrent. Without loss of generality, these three lines are ab, bc, cd . The first two lines intersect at b , and the last two lines intersect at c . Since two lines intersect at a unique point, and these two intersection points are distinct, these three lines cannot be concurrent.

- d. [3] Prove that the Fano Plane is the projective plane with the smallest number of points and lines (i.e., prove that no projective plane has fewer than 7 points or fewer than 7 lines, and prove that every projective plane with exactly 7 points and lines is equivalent to the Fano Plane).

Solution: Let a, b, c, d be points, no three of which are on the same line. As we saw earlier, connecting all 6 pairs of these points gives 6 distinct lines. So far, however, lines ab and cd ; ac and bd ; and ad and bc do not intersect. Thus, we must adjoin three more points e, f, g to the plane so that these three pairs of lines intersect. Now, there are 7 points and 6 lines, so we can only add one more line, which must go through e, f, g . This gives a projective plane of order 2, which is equivalent to the Fano Plane.

Now, note that conditions 1 and 2 for a projective plane are “dual” to each other, since they can be obtained from each other by switching the words “point(s)” and “line(s).” We just proved that the dual of condition 3 always holds as well. Therefore, if any statement is true for all projective planes, its dual statement will automatically be true as well.

3. We will now prove some more facts about finite projective planes (ones with finitely many points and lines).

- a. [3] Prove that, given any two distinct lines, there is a point not incident to either line.

Solution: Assume for contradiction that every point is on one of the two lines, call them l_1, l_2 . By the third condition, there are four points, no three of them on the same line. Thus, two of them must be on l_1 and not on l_2 , and the other two must be on l_2 and not on l_1 . If a, b are on l_1 and c, d are on l_2 , then the lines ad and bc must intersect at some point; this intersection point can't be on either line, else 3 of a, b, c, d would be collinear, a contradiction. Thus, we have a point not on either line.

- b. [2] Given a line l and a point p not incident to l , construct a bijection between points incident to l and lines incident to p . Use this and the previous result to conclude that, given any two distinct lines, there is a bijection between points incident to the first line and points incident to the second line.

Solution: We can map a point q on l to the line incident to both p and q . Conversely, we can map a line incident to p to its point of intersection with the line l . Now, given lines l_1 and l_2 , and a point p not on either, we have a bijection between points on l_1 and lines through p , and another bijection between lines through p and points on l_2 . Composing these bijections, we get a bijection between points on l_1 and points on l_2 .

- c. [2] Assume that a given projective plane has finitely many lines. Prove that if a point p is incident to $n + 1$ lines, then every line not incident to p is incident to $n + 1$ points.

Solution: By the previous part, there are as many points on a line not incident to p as there are lines incident to p , so every line not incident to p has $n + 1$ points.

- d. [3] Continuing to assume that there are finitely many lines, prove that there's an integer $n \geq 2$ such that every point is incident to $n + 1$ lines and every line is incident to $n + 1$ points. Conclude that the projective plane has $n^2 + n + 1$ points and the same number of lines. Such a finite projective plane will be said to have **order** n .

Solution: Let a, b, c, d be points no three of which are on the same line. Then each point is incident to at least 3 lines. Since there are finitely many lines, there's an integer $n \geq 2$ such that a is incident to $n + 1$ lines. By the previous part, every line not incident to a has $n + 1$ points. Also by the previous part, every line has the same number of points, so each line has $n + 1$ points (a line not incident to a exists, since bc is such a line). Now, if p is any point, then there is a line l not incident to p (if not, every line passes through p , contradicting the result of the first part), so by the previous part, p is incident to $n + 1$ lines, since l has $n + 1$ points.

Now, let p be any point. Then, the $n + 1$ lines through p contain all the points. Each line has $n + 1$ points, n of which are not equal to p . These $(n + 1)n$ points are all distinct since any two of these lines can only intersect at p . Thus, in addition to p , this gives $(n + 1)n + 1 = n^2 + n + 1$ points in total. By a dual argument, there are $n^2 + n + 1$ lines in total as well.

4. [8] A **degenerate projective plane** is a set of points and lines with an incidence relation that satisfies the first two conditions but fails the third, meaning that given any four points, at least three of them are incident to the same line. The empty plane, consisting of no points or lines, vacuously satisfies the first two conditions and fails the third. Prove that a nonempty degenerate projective plane is one of three following types:
- There is a line and no points, or there is a point and no lines.
 - There is a line l incident to all the points. There is a point p such that every other line (if there are other lines) is incident to just the point p .
 - There is a line l incident to all the points except one (call it p). Every other line (if there are any) is incident to just p and one point that is incident to l .

Solution: If there are no points, there can only be one line (since two lines would have to intersect at a point). Similarly, if there are no lines, there can only be one point. This gives the first case, so now assume that our plane has at least one point and one line.

If a line l is incident to all the points, then any other line must intersect l at the same point (for if l_1 intersects l at p_1 and l_2 does at p_2 , then the point of intersection of l_1 and l_2 can't be any point on l , contradicting l being incident to all the points). This gives the second case.

If there is only one point, then either there is just one line, giving a special case of the second or third cases, depending on if the point and line are incident, or there are at least 2 lines, and they all are incident to p , which is a special subcase of the second case. If there are two points, then the line between them is incident to all the points, and we have the case discussed above. If there are three points, they're either collinear, giving the second case, or they are not. In the latter case, there are points a, b, c , and lines ab, bc, ac . Any other line would have to intersect each of these three lines at exactly one point, which is impossible. Thus, we have a triangle, which is a special subcase of the third case.

Now assume there are at least four points. Since the third condition for a projective plane fails, for every four points, three of them are collinear. Let a, b, c, d be four points not all on the same line (if all points are on the same line, we have the second case), and assume a, b, c are on the same line l . We claim that any other point $p \neq a, b, c, d$ is also on line l . If not, then a, b, d, p are four distinct points, so three of them are collinear. Neither a, b, d nor a, b, p can be collinear, since their line would intersect l at two points. So without loss of generality, d, a, p are collinear, on line l_1 . But repeating

this argument for the four distinct points b, c, d, p gives that p, b, d or p, c, d are collinear, on line l_2 . But then l_1 and l_2 are distinct lines (since one contains a and the other contains b or c) that intersect at two points, d and p . This is a contradiction, so in fact, p is on line l . Thus, l is incident to every point except d , and the other lines just contain d and one point on l , giving the third case.

The Real Projective Plane (25 Points)

In this section, we will consider different but equivalent ways of turning the ordinary plane into a projective plane.

5. The motivation for the definition of a projective plane comes from the ordinary Euclidean plane, \mathbb{R}^2 . If we define points and lines to be ordinary points and lines in the real plane, and say that point is incident to a line if the point is on the line, then we have a geometry that satisfies conditions 1 and 3. Condition 2 also holds unless the two lines are parallel.

- a. [3] Determine the line through the points $(2, 3)$ and $(7, -4)$ in the plane. Write the line equation in the form $y = mx + b$. No proof is required.

Solution: The slope of the desired line is $\frac{3 - (-4)}{2 - 7} = -\frac{7}{5}$. Thus, the equation of the line is $y - 3 = -\frac{7}{5}(x - 2)$, which is equivalent to $y = -\frac{7}{5}x + \frac{29}{5}$.

- b. [2] Determine the point of intersection of the lines $y = 12x - 18$ and $y = \frac{1}{2}x + 5$. No proof is required.

Solution: Setting the y 's equal, we have $12x - 18 = \frac{1}{2}x + 5 \implies \frac{23}{2}x = 23 \implies x = 2 \implies y = 6$. Thus, the point of intersection is $(2, 6)$.

As we noted earlier, \mathbb{R}^2 is not quite a projective plane because parallel lines exist. We can remedy this by adding points at infinity and a line at infinity. For every possible slope m (where m can be any real number or even ∞ , for vertical lines), adjoin a point p_m so that p_m is incident to every line of slope m . Finally, adjoin a line at infinity that is incident to exactly the points at infinity p_m and no other points. It is clear that this gives a projective plane, which we will call the **extended real plane**.

6. Consider the Euclidean space \mathbb{R}^3 . Let P be the set of lines through the origin, and let L be the set of planes through the origin. Say that $p \in P$ and $l \in L$ are incident if $p \subseteq l$, i.e., if p is a line in the plane l .

- a. [4] Prove that P and L , under the given incidence relation, form a projective plane. We will call this projective plane the **real projective plane**.

Solution: Let l_1, l_2 be two distinct lines through the origin, and let $p_1 \in l_1$ and $p_2 \in l_2$ be non-origin points. Since the lines are distinct and intersect only at the origin, $0, p_1, p_2$ are not collinear and thus determine a unique plane through the origin containing both p_1 and p_2 , and thus containing l_1 and l_2 . So, every two distinct lines are incident to a unique plane.

Let p_1, p_2 be two distinct planes. Without loss of generality, assume p_1 is the xy plane (we can rotate the space to put one of the planes as the xy plane). Then, p_2 will be the set of solutions (x, y, z) to the equation $ax + by + cz = 0$, where $a, b, c \in \mathbb{R}$ are not all 0. The xy plane has equation $z = 0$, so the intersection of these planes is given by solutions to $ax + by = 0$ in the xy plane. Since p_2 is distinct from the xy plane, a and b are not both 0, so this gives a line through the origin in the xy plane. Therefore, there is a unique line incident to both planes.

Finally, there are four lines, no three of which are on the same plane. For example, take the three coordinate axes and the line through the origin and $(1, 1, 1)$.

Therefore, L and P form a projective plane.

- b. [6] In fact, the real projective plane is equivalent to the extended real plane. To show this, construct a bijection between the points of the extended real plane and lines in \mathbb{R}^3 through the

origin; and between lines of the extended real plane and planes in \mathbb{R}^3 through the origin; in such a way that incidence is preserved.

Hint: in \mathbb{R}^3 , consider a plane parallel to the xy plane, and use this plane as the base of an extended real plane.

Solution: Let p be the plane $z = 1$. If a line through the origin is not in the xy plane, then it will intersect p at a point. Thus, we can associate lines not in the xy plane to points in the real plane. If a line through the origin is in the xy plane, then associate it to the point p_m in the extended real plane, where m is the slope of this line. Now, if p' is a plane through the origin which is not the xy plane, then it will intersect the plane p in a line, so we can associate p' to this corresponding line in the real plane (which will also include a point at infinity). If p' is the xy plane, we can associate it to the line at infinity. This gives a bijection between points and between lines of the real projective plane and the extended real plane. If l is a line through the origin and p' is a plane through the origin containing l and not equal to the xy plane, then $l \cap p \subseteq p' \cap p$, so the corresponding point in the extended plane is incident to the corresponding line. If instead, p' is the xy plane, then the corresponding point is a point at infinity, and the corresponding line is the line at infinity, so they are still incident. Therefore, our bijections preserve incidence, so our two projective planes are equivalent.

7. Let S^2 be the unit sphere in \mathbb{R}^3 , let P be the set consisting of pairs of antipodal points, i.e., pairs of the form $\{x, -x\}$, for $x \in S^2$, and let L be the set of great circles on the sphere, which are circles whose center is the center of the sphere. Say that a pair of antipodal points is incident to a great circle if both the points are in the circle.

- a. [5] Show that this makes the sphere into a projective plane, which we will call the **projective sphere**.

Solution: If we have two non-antipodal points on the sphere, the unique great circle through them will also pass through both of their antipodes. Conversely, if we have two great circles, they will intersect in a pair of antipodal points. Finally, four pairs of antipodes, no three of which are on the same circle, are given by $\pm(1, 0, 0)$, $\pm(0, 1, 0)$, $\pm(0, 0, 1)$, $\pm(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$.

- b. [5] Show that the projective sphere is equivalent to the real projective plane, i.e., that we can map points and lines in one bijectively to points and lines in the other, in a way that preserves incidence.

Solution: Consider S^2 as a subspace of \mathbb{R}^3 . We can associate a line through the origin to the two antipodal points of intersection of that line with the unit sphere. Similarly, we can associate a plane through the origin to its intersection with the unit sphere, which will be a great circle. This is clearly a bijection, and if a line l is on a plane p , then since $l \subseteq p$, $l \cap S^2 \subseteq p \cap S^2$, so this bijection preserves incidence. Since S^2 together with P and L are equivalent to the real projective plane, they also make S^2 into a projective plane.

Collineations of Projective Planes (40 Points)

A **collineation** of a projective plane (P, L, I) is a function $f : P \cup L \rightarrow P \cup L$ such that f maps P bijectively onto P and L bijectively onto L (i.e., f permutes the points and lines), and such that f preserves the incidence relation, meaning that a point p and a line l are incident if and only if $f(p)$ and $f(l)$ are incident. The identity function, mapping all points and lines to themselves, is an obvious collineation. Also, if f, g are collineations, so are $f \circ g$ and f^{-1} .

8. [8] Prove that if a collineation fixes all points, it also fixes all lines. By duality, it would follow that any collineation fixing all lines would also fix all points.

Solution: Note that any line l must contain at least 2 points, since if not, then all lines pass through the one point (l cannot contain no points since, by the first problem, there are at least 4 lines and any pair of them intersect at a point), which contradicts there being four lines, no three of which are concurrent, a result of a previous problem. Since these two points are fixed, so is l , the unique line between them. Since l was arbitrary, the collineation fixes all lines.

9. A collineation f is called a **central** collineation, with center p , if f fixes the point p and fixes all lines incident to p . I.e., given a point $q \neq p$, the line pq is the same as the line $pf(q)$. We say f is **axial**, with axis l , if f fixes the line l and all points incident to l . I.e., if $l' \neq l$ is another line, then l' intersects l at the same point as $f(l')$ intersects l .

- a. [5] Prove that if a collineation of a projective plane has at least two distinct centers or at least distinct two axes, then it is the identity collineation.

Solution: Assume f is a collineation with two centers, meaning it fixes two distinct points a, b and all lines incident to either point. Let c be any other point. Then f fixes both of the lines ac and bc , and thus fixes their point of intersection, which is c . Since c was arbitrary, f fixes all points and it thus the identity.

Now assume f has two axes, l_1, l_2 . Let l_3 be any other line. Then the points $l_1 \cap l_3$ and $l_2 \cap l_3$ are fixed because they are both on axes. But then the line between these points, which is l_3 , is also fixed. Since l_3 was arbitrary, f fixes all lines and is thus the identity.

- b. [5] Prove that an axial collineation that fixes a point p not on the axis is also central, with center p . Similarly, prove that a central collineation that fixes a line l that doesn't contain the center is also axial, with axis l .

Solution: In the first case, the collineation fixes p . Let l be the axis and let l' be any line incident to p . Then the point $l' \cap l$ is fixed since it is on the axis. Since the distinct points p and $l' \cap l$ are fixed, so is the unique line between them, which is l' . Thus, any line incident to p is fixed, so p is a center of the collineation.

In the second case, the collineation fixes l . Let q be a point on l . Since the line pq is incident to p , it is fixed. Since both l and pq are fixed, so is their point of intersection, which is q . Thus, any point incident to l is fixed, so l is an axis of the collineation.

10. A collineation with center p and axis l is called a (p, l) -collineation.

- a. [6] Prove that a (p, l) -collineation is completely determined by its center, its axis, and by its effect on one point not on the axis and not equal to the center.

Solution: Let f, g be collineations with center p and axis l that send the point $q \neq p$ and not on l to the same point. We will show that $\phi = fg^{-1}$, which has center p and axis l and fixes q is the identity. We already know that p, q , and all points on l are fixed. Let r be a point that's not one of these, and let s be the point of intersection of the lines qr and l . Then s is fixed, since it's on the axis l . Also, since s is on line qr and is fixed by ϕ , s is also on the line $\phi(q)\phi(r) = q\phi(r)$, since q is fixed. Also, since the line pr is fixed (since it's incident to the center p), $\phi(r)$ is on this line. Thus, $\phi(r)$ is the point of intersection of the lines qs and pr . But we already know that r is on both lines, so $\phi(r) = r$. Since r was arbitrary, ϕ fixes all points and is thus the identity, so $f = g$.

- b. [6] Prove that a (p, l) -collineation of the Fano Plane is the identity if p is not incident to l . If p is incident to l , prove that there is always one nonidentity (p, l) -collineation.

Solution: Let a, b, c be the three points on l . Then, the four points a, b, c, p are all fixed. Since p is not on the same line as a, b, c , the three lines ap, bp, cp are all distinct. For each line, two points on it are fixed, so the third point is fixed as well. The three extra points on these three lines give the remaining three points, so all 7 points are fixed, and we have the identity collineation.

Now assume a, b, c are on l and b is the center. There are two lines besides l incident to b , and each contains two points besides b . We claim that the map that fixes a, b, c and switches the two extra points on both of the two extra lines is a collineation. We know that line l and the two extra lines are sent to themselves. We see that the two lines incident to a are switched, and the two lines incident to c are switched. Thus, lines are sent to lines and incidence is preserved, so we have a nonidentity collineation.

A projective plane is said to be (p, l) -transitive if for any points q_1, q_2 such that neither is the center or on the axis, and such that q_2 is on the same line as p and q_1 , there is a (p, l) -collineation that sends q_1 to q_2 . A projective plane is said to be **complete** if it is (p, l) -transitive for any point p and any line l .

11. [10] Prove that the real projective plane is complete. It may be helpful to use the equivalence with the extended real plane, the fact that rotations and translations are collineations in the extended real plane, and the fact that rotations fixing the origin are collineations in the real projective plane.

Solution: Without loss of generality, we can take l in the real projective plane to be the xy plane, since we can always rotate any plane through the origin to the xy plane, and rotations fixing the origin are collineations. Using the equivalence, this plane becomes the line at infinity in the extended real plane. First, assume that p is a point at infinity. Then q_1, q_2 are normal points which are collinear with p , meaning that the slope of the line q_1q_2 corresponds to p . Thus, we can perform a simple translation that moves q_1 to q_2 . Translations send lines to themselves, and they fix the line at infinity, as well as all its points (since lines are sent to lines of the same slope). Furthermore, the other lines incident to p , i.e., all the lines whose slope is the same as that of q_1q_2 , are fixed as well because the translation is along that slope. Thus, the translation provides the desired collineation.

Now assume that p, q_1, q_2 are normal points that are collinear. Without loss of generality, we can assume p is the origin, since we can perform a translation, which is a collineation, to move p to the origin. Since q_1 and q_2 are collinear with the origin, there exists a scalar c such that $q_2 = cq_1$. Then, the map which sends $v \mapsto cv$ takes $q_1 \mapsto q_2$, fixes all lines through the origin, and sends lines to lines of the same slope, thus also fixing the line and points at infinity. Thus, this expansion provides the desired collineation.