Time Limit: 60 mins.
Maximum Score: 125 points.
Instructions:

1. When a problem asks you to "compute" or "list" something, no proof is necessary. However, for all other problems, unless otherwise indicated, you must justify your answers.

2. You may freely assume results of a previous problem in proving later problems, even if you have not proved the previous result.

3. You may use both sides of the paper and multiple sheets of paper for a problem, but separate problems should be on separate sheets of paper. Label the pages of each problem as 1/2, 2/2, etc., in the upper right hand corner. Write your team ID at the upper-right corner of every page you turn in.

4. Partial credit may be given for partial progress on a problem, provided the progress is sufficiently nontrivial.

5. Throughout this round, unless stated otherwise, all groups are assumed to be finite, meaning they contain finitely many elements.

6. When you see a product of two group elements which represent permutations, symmetry operations, or group actions in general, for example gh, you should read it from **right to left**: first apply h, and then apply g. This is the same convention as for function composition.

7. *You may not use without proof results that are not discussed in this round.*

8. **Calculators are not allowed!**

Introduction: This power round deals with symmetry groups. We will begin by introducing groups, which are just mathematical objects that satisfy some rules. However, even though this definition is abstract, we will discuss a more intuitive view of groups as the set of symmetries of certain objects. The Zome kits provided may be helpful in visualizing the polyhedra and their symmetries. The general aim of this round is to provide a more visual and hands-on view of a subject that is often perceived as dry and abstract, and to give you a taste of the power of group theory. One of the things that group theory can show us is the equivalence between seemingly different types of objects, such as symmetries of a polyhedron and permutations of a set. Throughout this round, you will encounter many such equivalences.

Although different team members may work on different rounds, it is recommended that all team members at least read the first section, "Introduction to Groups," so that they are aware of the relevant definitions. Many of the sections can be solved independently, and those that require definitions and results from previous sections will say so in their introductions.

Introduction to Groups (20 Points)

We will begin with the definition of a group. Though it may seem a bit abstract, once we start discussing groups as symmetries of objects, the concepts should hopefully become more intuitive.

A **group**, $(G, \cdot)$ is a set, G, with a **binary operation** $\cdot$, that takes as inputs two elements of the set G and outputs another element in the set. I.e., if g and h are elements of the set G, then $g \cdot h$ is also in the set G. In addition, a group must satisfy the following properties:

a. **Associativity**: If g, h, and k are in G, then $(g \cdot h) \cdot k = g \cdot (h \cdot k)$.

b. **Identity Element**: There exists an element 1 in G such that, for every element g in the group, $g \cdot 1 = 1 \cdot g = g$. Note that this "1" is not the same as the number 1, but is a notation used to denote the identity element. However, since our groups will be written multiplicatively, and the number 1 is a multiplicative identity, this notation is a convenient way of representing the identity.

c. **Inverse Element**: For every element g in G, there exists an inverse g^{-1} such that $g \cdot g^{-1} = g^{-1} \cdot g = 1$, where 1 is the identity element. Note that both directions must equal 1 for g^{-1} to actually be an inverse.

For convenience, we will refer to a group as G instead of $(G, \cdot)$. An **element** of a group is just an element of its underlying set. In addition, we will usually omit the $\cdot$ when writing the binary operation: i.e., $g \cdot h$ will just be written as gh. We will also generally refer to the binary operation as multiplication.

A group G is said to be **abelian** if multiplication is commutative: if for all elements g, h in G, $gh = hg$. Note that groups are not abelian in general.

1. **[2]** Prove that the integers, $\mathbb{Z}$, form an abelian group under addition.

   Solution: The integers are closed under addition (the sum of two integers is an integer), addition is associative, 0 is the additive identity (not 1), and $-x$ is the additive inverse of x. Thus, the integers form a group under addition. Since addition of integers is commutative, this group is abelian.

Note that the integers are not a group under multiplication, since no integer besides 1 and -1 have a multiplicative inverse: 1 is clearly the identity, but there is no integer n such that $5n = 1$, so 5 is not invertible, and so $\mathbb{Z}$ is not a group under multiplication.

2. State why the following are not examples of groups:

   a. **[2]** The nonnegative integers, $\mathbb{N}_0$, under addition.

   Solution: 0 is still the additive identity. Since this set does not contain any negative integers, any nonzero integer in the set does not have an additive inverse.

   b. **[2]** The rational numbers, $\mathbb{Q}$, under multiplication.

   Solution: 1 is the multiplicative identity. 0 does not have a multiplicative inverse. However, if you remove 0, the resulting set does form a group under multiplication!

3. a. **[2]** Prove that the identity of a group is unique, and that every element has a unique inverse. In other words, show that if 1 and $1'$ are identities, then $1 = 1'$, and that if a and b are inverses of g, then $a = b$.

   Solution: Assume 1 and $1'$ are both identities. Then $1 * 1' = 1$, since $1'$ is an identity, and $1 * 1' = 1'$, since 1 is an identity. Thus, $1 = 1'$, so the identity is unique.

   Assume that g is a group element with inverses a and b. Then $agb = 1b = b$, since $ag = 1$, and $agb = a1 = a$, since $gb = 1$. Thus, $a = b$, so each element's inverse is unique.

   b. **[2]** Prove that if g and h are elements of a group G, then $(gh)^{-1} = h^{-1}g^{-1}$.

   Solution: $(gh)(h^{-1}g^{-1}) = gg^{-1} = 1$, and $(h^{-1}g^{-1})(gh) = h^{-1}h = 1$, so $(gh)^{-1} = h^{-1}g^{-1}$ (this inverse is unique by the previous part).

c. **[2]** Prove the cancellation laws: If g, h, and k are elements of a group, and if $gh = gk$, then $h = k$; Similarly, if $gk = hk$, then $g = h$. Conclude that if $gh = g$ or $hg = g$, then $h = 1$.

Solution: In the first case, multiply both sides by g^{-1} on the left to get the desired result. In the second case, multiply both sides by k^{-1} on the right. The last two are special cases of the first 2, where the third group element is 1.

4. **[3]** For an element g in G, g^k, where k is a positive integer, is the result of multiplying g by itself k times. In addition, $g^0 = 1$ and $g^{-k} = (g^{-1})^k$. The **order of a group element** g is the smallest positive integer k such that $g^k = 1$. If there is no such positive integer, we say the order of g is ∞. For example, 1 has order 1. Also, the **order of a group** is the number of elements in it. Prove that, if G has finite order, so does any element g in G.

   Solution: Since G is finite, the set $\{1, g, g^2, \ldots\}$ must eventually have the same group element at 2 different positions. Say $g^i = g^j$ for some $0 < i < j < \infty$. Then, multiplying both sides by g^{-i}, we get $1 = g^{j-i}$, so g has finite order.

5. **[2]** If g is an element of order n, prove that $g^{-1} = g^{n-1}$.

   Solution: $gg^{n-1} = g^n = 1$, so g^{n-1} is g's unique inverse.

6. **[3]** Prove that, if g is an element of a group G and $g^k = 1$, where k is a positive integer, then k is a multiple of the order of g.

   Solution: Let m be the order of g. By definition, $k \geq m$. We can thus write $k = qm + r$, where q and m are positive integers and $0 \leq r < m$. Then $1 = g^k = g^{qm+r} = (g^m)^q g^r = g^r$, since $g^m = 1$. But now we have that $g^r = 1$, where $r < m$. Since m by definition is the smallest positive integer such that $g^m = 1$, we must have $r = 0$, so $k = qm$, as desired.

Special Types of Groups

In the following sections, we will familiarize ourselves with some special types of groups and begin to see how they can be used to describe the symmetries of polyhedra.

Cyclic Groups (15 Points)

We will begin by introducing the simplest type of group: the **cyclic** groups. If C is a cyclic group, then every element c in C can be written as $c = g^k$, for some g in G. Such an element g is called a **generator** of the cyclic group (we will define generators for other groups later).

7. A **symmetry operation** on an object is a rotation, reflection, or possibly a combination of both, that keeps the object looking the same as it was originally. These operations form a group under composition, which is just applying one symmetry operation after another: If σ and τ are symmetry operations, then $\sigma\tau$ is a symmetry operation that represents applying τ and then σ. Note that we always read composed operations from right to left. The identity is just the operation of doing nothing, and the inverse of a symmetry operation is just the operation that reverses it, returning to the original position. Throughout this round, we will mostly be only considering the rotations, so you should only consider rotations unless told otherwise.

   a. **[2]** Consider a regular pentagonal pyramid, which you can build one with the provided Zome kits, using just blue struts. Describe the elements of the (rotational) symmetry group of the pentagonal pyramid. This will be the group of rotations of the pyramid that leaves it indistinguishable from its original state.

   Solution: The elements of this rotation group are clockwise rotations by 0, 72, 144, 216, and 288 degrees about the line between the apex and the center of the pentagonal base.

   b. **[2]** Show that this rotation group is cyclic, meaning that any rotation can be achieved by just applying some rotation some number of times.

   Solution: Any of the five rotations can be achieved by just applying the 72 degree rotation some number of times. Note that the identity can be considered to be this rotation applied 0 times.

c. **[2]** Show that this rotation group is equivalent to the group of integers mod 5 under addition, in the sense that each rotation can be associated to an integer mod 5 such that applying one rotation after another is associated with the sum of the associated integers of the two rotations.

Solution: We can associate the 0, 72, 144, 216, and 288 degree rotations to the integers 0, 1, 2, 3, and 4 mod 5. It is easy to see that this preserves both group operations, as desired.

8. Let C_n be the cyclic group generated by an element, g, of order n.

a. **[3]** Show that C_n's elements are exactly $\{1, g, g^2, \ldots, g^{n-1}\}$.

Solution: Since C_n is cyclic, every element can be written as g^k, for some integer k. By the division algorithm, we can write $k = qn + r$, with $0 \le r < n$, so $g^k = g^{qn+r} = (g^n)^q g^r = g^r$, since $(g^n)^q = 1^q = 1$. Thus, every element is equal to one of the elements in $\{1, g, g^2, \ldots, g^{n-1}\}$. These elements are all distinct: if $g^i = g^j$, where $i \le j$, then $g^{j-i} = 1$. But since $0 \le i, j < n$, $j - i < n$. This contradicts g having order n, unless $j - i = 0$, which means $i = j$. Thus, all the elements in this set are distinct, and so C_n as exactly n elements.

b. **[3]** What is the order of g^k, in terms of k and n? Given this, what must be true of k and n for g^k to be a(nother) generator?

Solution: Let $d = gcd(k, n)$. Then $(g^k)^{\frac{n}{d}} = (g^n)^{\frac{k}{d}} = 1$, since $g^n = 1$. Then, by a previous problem, the order of g^k, which we will now call m, must divide $\frac{n}{d}$. Since $(g^k)^m = g^{km} = 1$, and the order of g is n, then n divides km. If we let $a = \frac{n}{d}$, and let $b = \frac{k}{d}$, then a and b are relatively prime. We just saw that m must divide a. We also saw that n divides km, which means that ad divides bdm, meaning a divides bm. Since a and b are relatively prime, a must divide m. Thus, we have m and a are positive integers that divide each other, so $m = a = \frac{n}{d}$. Thus, for g^k to be another generator, we need k and n to be relatively prime.

c. **[3]** Prove that all cyclic groups are abelian.

Solution: In a cyclic group, any element can be written as a power of an element g. Let g^i and g^j be 2 such elements. Then $g^i g^j = g^{i+j} = g^{j+i} = g^j g^i$. Thus, any two elements commute with each other, so cyclic groups are abelian.

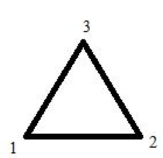
It turns out that any finite cyclic group C_n is equivalent to the rotational symmetry group of a regular $n$gonal pyramid, and also to the group of integers mod n under addition.

Dihedral Groups (10 Points)

We will now discuss a new type of group, called the **dihedral** groups. D_{2n}, the dihedral group of order $2n$, is the symmetry group of a regular $n$gon, including both rotations and reflections (this is one case where we will consider reflections). We can see that there are n ways to rotate the $n$gon (including the identity operation), and that there are n reflection axes, giving a total of $2n$ symmetry operations as elements of D_{2n}. Let r be the element of D_{2n} that represents a clockwise rotation by $\dfrac{360}{n}$ degrees, and let s represent reflection over a fixed axis. It is not difficult to see that r has order n and s has order 2, and that r and s generate the entire group.

9. **[2]** Consider an equilateral triangle with vertices labeled 1, 2, and 3 in clockwise order, with 1 pointing up, and let s represent reflection about the vertical axis. Draw the triangle after the symmetry operation sr is performed, i.e., after first applying r and then applying s (by convention, operations are applied from right to left, similar to function composition). It may help to build a triangle from the blue Zome struts to help you visualize the symmetry group.

Solution: We rotate clockwise by 120 degrees and then flip the triangle across the vertical axis, resulting in the following:



10. a. **[5]** Prove that sr^k has order 2 for all $0 \le k \le n-1$, and conclude that $sr^k = r^{-k}s$.

Solution: If we start with an $n$gon with vertices labeled 1 to n in clockwise order, with 1 pointing up, and let s represent reflection across the vertical axis, then it's not too hard to see that, since sr^k sends 1 to the vertex originally occupied by vertex $n-k+1 \bmod n$, and makes the vertex labels go counterclockwise, that this operation is a reflection about the line through the center that bisects the line between vertex 1 and vertex $n-k+1 \bmod n$. Any reflection applied twice results in the identity operation, so sr^k has order 2, meaning that it's its own inverse. Thus, $sr^k = (sr^k)^{-1} = r^{-k}s$.

b. **[3]** Show that D_{2n} is not abelian when $n > 2$.

Solution: Since $sr = r^{-1}s$, and $r \neq r^{-1}$ (since $n > 2$, and r has order n), D_{2n} is not abelian when $n > 2$.

Note that, though D_{2n} can be seen as the full symmetry group of a regular $n$gon, it can also be seen as the rotation group of an $n$gonal prism. Thus, the cyclic and dihedral groups, the simplest groups, can be thought of as the rotation groups of pyramids and prisms, the simplest polyhedra. To describe the rotation groups of more complicated polyhedra, we will need to introduce two new types of groups.

Symmetric Groups (20 Points)

S_n, the **symmetric** group on n letters, is the group of permutations of n objects. Formally, a **permutation** is a function from from the set of first n positive integers to itself such that the function does not send any two integers to the same integer. Intuitively, you can think of a permutation as a "reordering" of these n integers. Since permutations can be thought of as functions, the group operation will be function composition, i.e., applying one permutation after another. Thus, S_n has order $n!$, which is the number of ways to permute the n integers.

We will now introduce a very useful notation, known as cyclic notation, for representing elements of S_n. In this notation, elements are written as products of disjoint cycles, where each number in a cycle is sent to the next number, with the last number being sent to the first. For example, if we consider S_3 acting on the ordered set $\{1, 2, 3\}$, the element of S_3 that sends 1 to 2, 2 to 1, and 3 to 3, resulting in $\{2, 1, 3\}$, can be written in this notation as (12)(3). Cycles of length 1 can be omitted, giving the more concise (12). In a similar fashion, (123) represents sending 1 to 2, 2 to 3, and 3 to 1, resulting in $\{2, 3, 1\}$. The identity permutation is still written as 1. Note that (12) = (21) and (123) = (231) = (312). One can see after some thought that this cycle decomposition is unique up to the ordering of the cycles and circular permutations of the numbers in each cycle. By convention, we write the smallest number in the cycle first, and we will order the cycles by the first number in each. From now on, all elements of symmetric groups will be written in cyclic notation.

11. **[2]** Compute the inverses of the following permutations: (1 2), (1 2 3 4 5), and (1 2)(3 4 5)(6 8 7 9). No proof is required.

Solution: The respective inverses are (1 2), (1 5 4 3 2), and (6 9 7 8)(3 5 4)(1 2).

12. **[2]** Compute the following product of permutations: (1 2 3)(2 3 4)(2 1 4). Recall that permutations are applied from the right to the left. No proof is required.

Solution: The product is (1 3 4).

13. a. **[2]** Using cyclic notation, write down all the elements of S_3.

Solution: The elements of S_3 are 1, (1 2), (2 3), (1 3), (1 2 3), and (1 3 2).

b. **[3]** Prove that S_3 is equivalent to D_6, the symmetry group of an equilateral triangle.

Solution: Note that each both S_3 and D_6 have 6 elements. In addition, note that an element of D_6 permutes the vertices of an equilateral triangle. If we start with a triangle whose vertices are labeled 1, 2, and 3, clockwise, then 1, r, and r^2 give all clockwise permutations of the vertices, and s, sr, and sr^2 give all counterclockwise permutations of the vertices. Thus, since D_6 gives all permutations of a triangle's vertices, it's equivalent to S_3.

c. **[3]** Prove that S_n is not abelian for $n > 2$.

Solution: Since $n > 2$, the elements (1 2) and (1 3) are in S_n. Note that (1 2)(1 3) switches 1 and 3 and then switches 1 and 2 (reading the product from right to left), so the resulting permutation is (1 3 2), since 3 occupies 1's former position, 2 occupies 3's former position, and 1 occupies 2's former position. However, (1 3)(1 2) results in (1 2 3) by similar reasoning, so $(12)(13) \neq (13)(12)$, so S_n is not abelian when $n > 2$.

14. **[3]** Show that, when written as the product of disjoint cycles, the order of an element of a symmetric group is the least common multiple of the lengths of all of its cycles. You can assume that disjoint cycles commute, i.e., if σ and τ are disjoint cycles, then $\sigma\tau = \tau\sigma$.

Solution: The order of a single cycle is clearly just the length of the cycle. Since disjoint cycles commute, raising a permutation to some power is the same as raising each cycle to that power. So for a permutation σ, if $\sigma^n = 1$, then n must be a multiple of every cycle length. Thus, the order of σ is the least common multiple of the lengths of its cycles.

15. **[5]** Prove that S_n is generated by all of its 2-cycles, i.e., its transpositions. This means that you can get any permutation in S_n by composing some number of transpositions.

Solution: Note that an k-cycle can be written as a product of transpositions: $(a_1a_2\dots a_k) = (a_1a_k)(a_1a_{k-1})\cdots(a_1a_3)(a_1a_2)$. Thus, any product of disjoint cycles, and thus any element of S_n, can be written as a product of transpositions, so transpositions generate S_n.

Alternating Groups (20 Points)

Note: It is strongly recommended that you at least read the section on symmetric groups before doing these problems.

A_n, the **alternating** group on n letters, is the group of even permutations of S_n. Recall that any element of S_n can be written as a product of transpositions. A_n then consists of those elements of S_n that can be written as a product of an even number of transpositions. Such elements are called **even permutations**, while elements of S_n that can be written as a product of an odd number of transpositions are called **odd permutations**.

16. a. **[8]** Prove that a permutation σ in S_n cannot be both even and odd: that is, show that σ cannot be written both as a product of an even number of transpositions and a product of an odd number of transpositions. Possible hint: begin by proving that 1 is even and not odd.

Solution: Clearly, 1 is even, since it can be written as the (empty) product of 0 transpositions. Alternatively, if you don't buy that 0 is even, then note that 1=(1 2)(1 2), and 2 is certainly even, so 1 is therefore even. To show that 1 is not odd, assume that $1 = t_1t_2\cdots t_j$, where j is odd and the t_i's are transpositions. We seek to show that 1 can be written as a product of $t-2$ transpositions: let x be a number that appears in t_1. Obviously, x must appear in at least 1 other transposition, since the product of these transpositions is 1, meaning that x must move back to its original position. So say x first appears again in t_i, where $1 < i \leq j$. Let $t_i = (xy)$ and consider t_{i-1}. Unless $i = 2$ (which we will soon consider), t_{i-1} does not contain x since we assumed t_i contained the first x after the 1st transposition. So either t_{i-1} shares y with t_i or it is disjoint with t_i. In the first case, note that $(yz)(xy) = (xz)(yz)$. In the second case, we see that $(ab)(xy) = (xy)(ab)$. In either case, we can move the cycle containing x closest to t_1 without changing the number of x's in the product. Repeating this, we eventually move the cycle containing x to the 2nd position, making the first 2 cycles $(xa)(xb)$. If $a = b$, the the 2 cycles cancel, and we've reduced our product to $j-2$ transpositions. Otherwise, $(xa)(xb) = (axb) = (bx)(ba)$. In this case, we've reduced the number of x's in our product by 1. There must be another x in the product, so we can do this again. Eventually, the final 2 x's must cancel each other out, reducing the product to $j-2$ transpositions. Thus, we can keep reducing our product by 2 transpositions each time, until we get 1 as the product of 1 transposition, which is clearly impossible. Thus, 1 is not an odd permutation.

This means that a permutation σ cannot be both odd and even: If we assume for sake of contradiction that $\sigma = t_1t_2\cdots t_{2n} = T_1T_2\cdots T_{2n+1}$, then, since the inverse of any transposition is itself, $1 = T_1T_2 \cdots T_{2n+1}t_{2n} \cdots t_2t_1$. We have thus written 1 as the product of an odd number of transpositions, which we just showed is impossible. Thus, σ cannot be both even and odd.

b. **[4]** Prove that the product of two even and of two odd permutations is even, while the product of an even and an odd permutation is odd. Conclude that the order of A_n is $\frac{n!}{2}$ for $n > 1$.

Solution: Consider two permutations σ and τ. If σ can be written as the product of s transpositions, and τ can be written as the product of t transpositions, then $\sigma\tau$ can be written as the product of $s + t$ transpositions. If s and t are both even or both odd, then $s + t$ is even; if one of s and t is even and the other is odd, then $s + t$ is odd. Thus, the product of two even or of two odd permutations is even while the product of two permutations of opposite parity is odd.

Consider the set of elements of S_n and then consider the set of elements in S_n multiplied by the odd permutation (1 2). Since this operation is invertible, both sets contain every element of S_n exactly once. However, every odd permutation is sent to an even permutation and vice-versa. Since both sets have the same number of even permutations and the same number of odd permutations as each other, it follows that S_n contains an equal number of odd and even permutations, which means the order of A_n, the group of even permutations, is $\frac{n!}{2}$.

c. **[3]** Show that, when written in cyclic notation, the elements of A_n are those elements of S_n with an even number of cycles of even length. Use this fact to list the elements of A_4.

Solution: Note that a cycle $(a_1a_2 \cdots a_k)$ can be written as the following product of transpositions: $(a_1a_k)(a_1a_{k-1}) \cdots (a_1a_3)(a_1a_2)$. This product contains $k - 1$ transpositions, so an even-length cycle represents an odd permutation and vice-versa. Thus, for a permutation to be odd, it must have an even number of cycles of even length.

Therefore, the elements of A_4 are 1, (1 2 3), (1 3 2), (1 2 4), (1 4 2), (1 3 4), (1 4 3), (2 3 4), (2 4 3), (1 2)(3 4), (1 3)(2 4), and (1 4)(2 3).

17. **[5]** Prove that A_n is generated by all of its 3-cycles. This means that any permutation in A_n can be achieved by just composing some number of 3-cycles.

Solution: Any element of A_n can be written as the product of an even number of transpositions, so it suffices to show that any product of two transpositions can be written as a product of 3-cycles. We can assume these transpositions are not equal, otherwise we can cancel them. Note that $(xy)(xz) = (xzy)$ and $(xy)(zw) = (xwz)(xyz)$. Also, 1 can be written as the empty product of 0 3-cycles. Thus, any element in A_n can be written as the product of 3-cycles.

Although Symmetric and Alternating groups may currently seem unrelated to symmetry groups, we will see in the next section that the rotation groups of the regular polyhedra are just these types of groups.

Regular Polyhedra and Their Rotation Groups (40 Points)

Note: It is strongly recommended that you at least read the sections on symmetric and alternating groups before doing the problems in the rest of this round.

The regular polyhedra, also known as the Platonic Solids, are polyhedra whose faces are all identical regular polygons, and whose vertices are all identical, having the same number of these polygons meeting at each vertex. The 5 Platonic Solids are listed below. These are the only five Platonic Solids, though we will not prove this here.

The **tetrahedron** has 3 triangles around each vertex. It has 4 faces, 4 vertices, and 6 edges.

The **cube** has 3 squares around each vertex. It has 6 faces, 8 vertices, and 12 edges.

The **octahedron** has 4 triangles around each vertex. It has 8 faces, 6 vertices, and 12 edges.

The **dodecahedron** has 3 pentagons around each vertex. It has 12 faces, 20 vertices, and 30 edges.

The **icosahedron** has 5 triangles around each vertex. It has 20 faces, 12 vertices, and 30 edges.

It is highly encouraged that you construct all five of them with the provided Zome kits. The tetrahedron and octahedron can be constructed with just green struts, and the other three can be constructed with just blue struts.

Every polyhedron has a **dual** polyhedron, which can be constructed by placing a point at the center of each of the original polyhedron's faces, and connecting each resulting point with the resulting points of all neighboring faces of the original polyhedron. It is straightforward to see that the tetrahedron is self-dual, that the cube and octahedron are dual to each other, and that the icosahedron and dodecahedron are dual to each other.

18. **[5]** Prove that the rotation group of a tetrahedron is equivalent to A_4 by considering its effect on the vertices of the tetrahedron, and by computing its order.

    Solution: There are 4 3-fold axes to rotate around (they pass from each vertex to the center of the opposite face), and each axis permits 2 nonidentity rotations, giving 8 rotations. In addition, there are 3 2-fold axes (passing through the midpoints of opposite pairs of edges), giving 3 nonidentity rotations. Adding the identity, we get a total of 12 rotations in the rotation group. Note that these are the only rotations since any rotation is completely determined by where it sends two of the vertices: the first vertex has 4 choices, and the second has 3, giving a total of 12 rotations. Also note that, if we label the vertices of a tetrahedron as 1, 2, 3, and 4, then a rotation about a 3-fold axis is equivalent to a 3 cycle (since any such rotation rotates the 3 vertices of one face of the tetrahedron while fixing the 4th vertex), and a rotation about a 2-fold axis is equivalent to a pair of transpositions (since each such rotation switches the vertices on each of the edges the rotation axis passes through). Thus, the tetrahedral rotation group is equivalent to A_4.

Group Actions

A group G is said to **act** on a set S if, for each element g of G, g acts like a function from S to S, sending elements of S to elements of S. The only restrictions are that, if s is an element of S, $1(s) = s$ (the identity sends s to itself) and if g and h are elements of G, $h(g(s)) = (hg)(s)$, which means that applying g and then h to s gives the same result as applying hg to s. From now on, we will write $g(s)$ as $g \cdot s$ for brevity.

Examples of group actions are S_n acting on the set $\{1, \ldots, n\}$ by permutation, or D_{2n} acting on the vertices of an $n$gon by symmetry operations. We will soon study the action on the Platonic Solids by their rotation groups.

The **orbit** of an element s of S under a group G, denoted O_s. is the set of all elements of S that s can be sent to by elements of G. In other words, $O_s = \{g \cdot s \mid g \in G\}$.

The **stabilizer** of s, denoted as G_s is the set of all elements of G that send s to itself. In other words, $G_s = \{g \in G \mid g \cdot s = s\}$.

19. **[2]** Show that, if G acts on S, and a and b are elements of S such that $b = g \cdot a$ for some g in G, then $a = g^{-1} \cdot b$.

    Solution: From the properties of a group action, we have that $g^{-1} \cdot b = g^{-1} \cdot (g \cdot a) = (g^{-1}g) \cdot a = 1 \cdot a = a$.

20. a. **[3]** Find the number of elements in the orbit and stabilizer of the center of one of a cube's faces under its rotation group. Here, the rotation group of the cube acts on the face centers.

    Solution: Each face center can be sent to 6 face centers, since a cube has 6 faces. Each face center is fixed by the 4 rotations (including the identity) whose rotation axis is the one passing though the face center and that of the opposite face. Thus, the orbit has 6 elements and the stabilizer has 4 elements.

    b. **[3]** Repeat part a, using the midpoint of an octahedron's edge.

    Solution: An octahedron has 12 edges, so the orbit has 12 elements. The identity rotation and the 180 degree rotation about the line passing through the midpoints of an opposite pair of edges fix the midpoints of both edges, so the stabilizer has 2 elements.

    c. **[8]** Orbit-Stabilizer Theorem: Prove that if G acts on S, and s is an element of S, then $|G| = |O_s| \cdot |G_s|$.

    Solution: Let x be any element in the orbit of s and let g be an element of G that takes s to x. Then for every element h in the stabilizer of s, the element gh also sends s to x, since

$(gh) \cdot s = g \cdot (h \cdot s) = g \cdot s = x$. In addition, if u is another element that sends s to x, then $g^{-1}u$ is in the stabilizer of s since $(g^{-1}u) \cdot s = g^{-1}(u \cdot s) = g^{-1} \cdot x = s$. Thus, since for every element in the stabilizer of s, there is an element that sends s to x, and for every element that sends s to x, there is an element of the stabilizer of s, the stabilizer must be equal in size to the set of elements that send s to x. This means that the same number of group elements send s to any element of the orbit of s, so the size of the group is equal to the size of the stabilizer of s times the size of the orbit of s, thus proving the Orbit-Stabilizer Theorem.

d. **[6]** Use the Orbit-Stabilizer Theorem and your answers to parts a and b to find the order of the cubic and octahedral rotation groups. In addition, prove that both of these rotation groups are equivalent to S_4 by finding a set of 4 objects in the cube and octahedron that can be completely permuted by the respective rotation groups. (You needed to compute the order of the rotation group first to make sure that it wasn't a larger group that just contained S_4 as a subgroup).

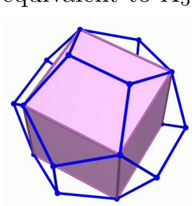
Solution: For the cube's face center, the orbit has 6 elements and the stabilizer has 4, so the order of the cubic rotation group is 24. Similarly, for the octahedron's edge midpoint, the orbit has 12 elements and the stabilizer has 2, so the order of the octahedral rotation group is again 24. To show that both rotation groups are equivalent to S_4, consider the 4 lines passing between opposite vertices in the cube and between opposite face centers in the octahedron. Since S_4 is generated by its transpositions, we only need to show that any 2 of these lines can be switched while fixing the other 2. In the cube, this is achieved by rotating by 180 degrees about a line connecting opposite edge midpoints: the diagonals attached to those 2 opposite edges switch places, while the other 2 diagonals both just rotate 180 degrees about their centers, remaining fixed as a whole. Thus, all 24 permutations of the cube's 4 diagonals can be achieves, and since the order of the cubic rotation group is 24, the group is equivalent to S_4. The octahedral rotation group can also be seen to be equivalent to S_4 by a similar argument: in this case, to switch 2 lines between 2 pairs of opposite face centers, rotate 180 degrees about the line connecting the midpoints of the 2 edges that connect the pairs of adjacent faces (if we choose 2 pairs of opposite faces in an octahedron, the 4 chosen faces also come in 2 pairs of adjacent faces).

You have shown that the cubic and octahedral rotation groups are equivalent. This turns out to be a general property of dual polyhedra, which means that the dodecahedral and icosahedral rotation groups are also equivalent. This is because the vertices of one polyhedron correspond to the faces of its dual and vice versa, while the edges in one polyhedron correspond to the edges in its dual.

21. **[3]** Use the Orbit-Stabilizer Theorem to determine the order of the icosahedral/dodecahedral rotation group.

    Solution: A face center of an icosahedron has an orbit size of 20, since it can be sent to any of the 20 face centers by the rotation group. Its stabilizer has size 3, consisting of the three rotations, including the identity, around the 3-fold axis connecting the face center with the opposite face center. Thus, the order of the icosahedral, and thus dodecahedral, rotation group is 60.

22. **[10]** Consider a dodecahedron. As shown in the figure below, you can inscribe a cube in the dodechaedron by drawing some of the diagonals of the dodecahedron's pentagonal faces. By drawing all 5 diagonals of all 12 pentagonal faces, 5 cubes can be inscribed in a dodecahedron. Show that there is some rotation that achieves each even permutation of these 5 cubes and, combined with your knowledge of the order of this group, conclude that the dodecahedral, and thus the icosahedral, rotation group is equivalent to A_5.

    Solution: We can see that the dodecahedral rotation group acts on these 5 cubes, permuting them. Note that with this construction, any 2 cubes share 2 vertices with each other, and these 2 vertices are opposite vertices of the dodecahedron. Thus, a 120 degree clockwise or counterclockwise rotation around the line connecting these 2 vertices will fix the 2 cubes and the other 3 cubes will be sent to each other, which corresponds to a 3-cycle in A_5. Since the 3-cycles generate A_5, we now know that the dodecahedral (and icosahedral) rotation group at least contains all elements of A_5. Since there is no way to switch 2 cubes while fixing the other 3, only even permutations of the cubes are possible. Alternatively, we can note that the order of this rotation group is 60, since we counted the rotations with conjugacy classes. Thus, since A_5 already has order 60, the icosahedral rotation group is equivalent to A_5.

Conclusion (0 Points)

We have now seen polyhedra whose rotation groups are the cyclic groups, the dihedral groups, and the Platonic groups (tetrahedral, octahedral, icosahedral). It turns out that these are the only finite rotation groups in 3 dimensions. We have actually developed all of the tools needed to prove this result, so if anyone is interested in the proof, it will be provided in the solutions. Feel free to try to prove it yourself, though no points will be given for it.

Solution: Let G be a finite rotation group in three dimensions. Letting G act on the unit sphere centered at the origin, we can see that each nonidentity rotation fixes exacty two antipodal points (the endpoints of the resulting axis of rotation). We will denote the set of those points fixed by some nonidentity element of G as P. We see that G sends fixed points to fixed points: if the point p is fixed by the nonidentity rotation h, and if g is an arbitrary element of G, then $g(p)$ is fixed by ghg^{-1}, which is not the identity since h isn't.

Now, since each nonidentity element of G fixes exactly one pair of antipoidal points, and letting G_p be the stabilizer of the point p, we get that $|G| - 1 = \frac{1}{2}\sum_{p\in P} |G_p| - 1$. We exclude the identity from all the stabilizers, and divide by 2 since each nonidentity element stabilizes 2 points. Now, letting O_p be the orbit of p under G, we see that the Orbit-Stabilizer Theorem gives $|G| - 1 = \frac{1}{2}\sum_{p\in P} \frac{|G|}{|O_p|} - 1$.

Note that, within an orbit, $\sum_{p\in O_p} \frac{|G|}{|O_p|} - 1 = |O_p|\left(\frac{|G|}{|O_p|}\right) - |O_p| = |G| - |O_p|$. Thus, the above sum can be rewritten to sum over the orbits: $|G| - 1 = \frac{1}{2}\sum_{O} |G| - |O|$. Rearranging and dividing by $|G|$ (and recalling the Orbit-Stabilizer Theorem), we get that $2 - \frac{2}{|G|} = \sum_{O}\left(1 - \frac{1}{|G_p|}\right)$, for some $p \in O$. If we let $a_1, \ldots, a_r$ be the sizes of the stabilizers of distinct orbits, then this becomes $2 - \frac{2}{|G|} = \sum_{i=1}^{r}\left(1 - \frac{1}{|a_i|}\right)$, where each a_i divides $|G|$ and is a positive integer.

Note that $a_i > 1$, since every fixed point is fixed by the identity and, by definition, at least one nonidentity element. Thus, each term in the sum on the right is at least $\frac{1}{2}$, and since the left hand side is less than 2, r must be less than 4. We will now check all the cases:

$r = 1$: Then $2 - \frac{2}{|G|} = 1 - \frac{1}{a_1}$. Since the right hand side is less than 1, $|G|$ (and a_1) must be 1, giving the trivial group, also known as the cyclic group of order 1, C_1. Note that in this case, the set of fixed points is empty, so the right hand side of the original equation was infact an empty sum, and r is actually equal to 0. This group is the rotation group of an object with absolutely no rotational symmetry.

$r = 2$: Then $2 - \frac{2}{|G|} = 2 - \frac{1}{a_1} + \frac{1}{a_2}$, which means that $\frac{2}{|G|} = \frac{1}{a_1} + \frac{1}{a_2}$. Since neither a_1 nor a_2 can be greater than $|G|$, we must have that $a_1 = a_2 = |G| = n$. Thus, each point is stabilized by the whole group, and there are exactly two such points, necessarily antipodal. Thus, G is a group of rotations perpendicular to the line through those two points, giving the cyclic group of order n, C_n, which we have seen is the rotation group of a regular $n$gonal pyramid (if $n = 2$, we can take a non-square rectangular pyramid, and if $n = 3$, we must make sure the base triangle is not congruent to the lateral triangles, or we would have a regular tetrahedron).

$r = 3$: Then $2 - \frac{2}{|G|} = 3 - \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3}$, which can be rearranged as $1 = \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} - \frac{2}{|G|}$. Note that this means $\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} > 1$., which means that at least one of the a_i's, without loss of generality, a_1, must equal 2. Now assume the a_i's are in ascending order. If $a_2 = 2$ as well, and $a_3 = n$, then $|G|$ must be $2n$. Thus, a_3's orbit is of size 2, and since antipodal points have the same orbit size, this orbit must consist of a pair of antipodal points, which are stabilized by n of G's elements that act as rotations around the line connecting the two points. There is also an element of G that switches these 2 points (since they're in each other's orbits), corresponding to reflections on the plane perpendicular to the center of the line between the antipodal points. This thus gives the dihedral group of order $2n$, D_{2n}. Note that D_{2n} is the rotation group of a regular $n$gonal prism (if $n = 2$, take a rectangular prism whose length, width, and height are all different, and if $n = 4$, make the height of a different length than the square base, so that the prism isn't a cube).

Since $\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} > 1$, and $a_1 = 2$, we cannot have both a_2 and a_3 be greater than 3, so a_2=3. The remaining choices for a_3 are then 3, 4, and 5. We will consider all of these cases for (a_1, a_2, a_3):

(2, 3, 3): This forces $|G| = 12$, and also forces $|P| = 14$, with orbits of sizes 4, 4, 6. These correspond to the vertices, face center, and edge midpoints of a regular tetrahedron.

(2, 3, 4): This forces $|G| = 24$, and also forces $|P| = 26$, with orbits of sizes 6, 8, and 12, which correspond to the vertices, face centers, and edge midpoints of a regular octahedron.

(2, 3, 5): This forces $|G| = 60$, and also forces $|P| = 62$, with orbits of sizes 12, 20, and 30, which correspond to the vertices, face centers, and edge midpoints of a regular icosahedron.

Thus, the only finite rotation groups in three dimensions are the cyclic, dihedral, and Platonic rotation groups. ■