GEOMETRY ROUND

1. Consider a regular hexagon with an incircle. What is the ratio of the area inside the incircle to the area of the hexagon?

2. Regular hexagon $ABCDEF$ has side length 2 and center O. The point P is defined as the intersection of AC and OB. Find the area of quadrilateral $OPCD$.

3. Consider an isosceles triangle $ABC(AB = BC)$. Let D be on BC such that $AD \perp BC$ and O be a circle with diameter BC. Suppose that segment AD intersects circle O at E. If $CA = 2$ what is CE?

4. A cylinder with length l has a radius of 6 meters, and three spheres with radii $3, 4$, and 5 meters are placed inside the cylinder. If the spheres are packed into the cylinder such that l is minimized, determine the length l.

5. In a 100-dimensional hypercube, each edge has length 1. The box contains $2^{100} + 1$ hyperspheres with the same radius r. The center of one hypersphere is the center of the hypercube, and it touches all the other spheres. Each of the other hyperspheres is tangent to 100 faces of the hypercube. Thus, the hyperspheres are tightly packed in the hypercube. Find r.

6. Square $ABCD$ has side length 5 and arc BD with center A. E is the midpoint of AB and CE intersects arc BD at F. G is placed onto BC such that FG is perpendicular to BC. What is the length of FG?

7. Consider a parallelogram $ABCD$. E is a point on ray $\overrightarrow{AD}$. BE intersects AC at F and CD at G. If $BF = EG$ and $BC = 3$, find the length of AE.

8. Semicircle O has diameter $AB = 12$. Arc $AC = 135°$. Let D be the midpoint of arc AC. Compute the region bounded by the lines CD and DB and the arc CB.

9. Let ABC be a triangle. Construct points B' and C' such that ACB' and ABC' are equilateral triangles that have no overlap with $\triangle ABC$. Let BB' and CC' intersect at X. If $AX = 3$, $BC = 4$, and $CX = 5$, find the area of quadrilateral $BCB'C'$.

10. Consider 8 points that are a knight's move away from the origin (i.e., the eight points $\{(2, 1), (2, -1), (1, 2), (1, -2), (-1, 2), (-1, -2), (-2, 1), (-2, -1)\}$). Each point has probability $\frac{1}{2}$ of being visible. What is the expected value of the area of the polygon formed by points that are visible? (If exactly 0, 1, 2 points appear, this area will be zero.)

P1. Let ABC be a triangle. Let r denote the inradius of $\triangle ABC$. Let r_a denote the A-exradius of $\triangle ABC$. Note that the A-excircle of $\triangle ABC$ is the circle that is tangent to segment BC, the extension of ray AB beyond B and the extension of AC beyond C. The A-exradius is the radius of the A-excircle. Define r_b and r_c analogously. Prove that

$$\frac{1}{r} = \frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c}.$$

P2. Let ABC be a fixed scalene triangle. Suppose that X, Y are variable points on segments AB, AC, respectively such that $BX = CY$. Prove that the circumcircle of $\triangle AXY$ passes through a fixed point other than A.