DISCRETE ANSWER KEY

1. 2184
2. 64
3. 96
4. 1815
5. $\dfrac{80}{243}$
6. $\dfrac{1}{18}$
7. $\{1, 2, 3\}$
8. 46
9. $\dfrac{191}{280}$
10. $\dfrac{53}{2}$

P1. Let a simple polygon be defined as a polygon in which no consecutive sides are parallel and no two non-consecutive sides share a common point. Given that all vertices of a simple polygon P are lattice points (in a Cartesian coordinate system, each vertex has integer coordinates), and each side of P has integer length, prove that the perimeter must be even.

Solution 1: Let (x_i, y_i) be the coordinates of the ith vertex of the polygon. Notice that $x \equiv x^2 \mod 2$, so

$$L = \sum_i \sqrt{(x_i - x_{i+1})^2 + (y_i - y_{i+1})^2} \equiv \sum_i (x_i - x_{i+1})^2 + (y_i - y_{i+1})^2 \equiv \sum_i x_i - x_{i+1} + y_i - y_{i+1} = 0 \mod 2$$

- 2 points for recognizing that the parity of x is the same as the partity of x^2 (in some form).
- 2 points for using the fact that $\sum \Delta x_i = 0$ so the number of odd Δx_i's is even. (or equivalent statement) wher $\Delta x_i = x_{i+1} - x_i$
- 2 points for complete solution.

Solution 2: Since the vertices are on integer coordinates, we know that each of the side lengths is in the form $\sqrt{a^2 + b^2}$, where a, b are integers. We know that an edge is even if $\sqrt{a^2 + b^2}$ is even, or in other words, both a, b are even. We get that an edge is odd if exactly one of them is odd. Thus, for a segment $(x_0, y_0) \to (x_1, y_1), a = |x_0 - x_1|, b = |y_0 - y_1|$, if both a, b are even, then the length is even. Otherwise, if exactly one of those differenes is odd, the edge length is odd. Thus, if the parity of x changes from x_0 to x_1, then a is odd, and similarly for y.Thus, if we consider the parity of the sum of the coordinates,we can see that if it changes, then our edge is odd, otherwise it is even. Now, consider the polygon $(x_0, y_0) \to (x_1, y_1) \to \ldots \to (x_k, y_k) \to (x_0, y_0)$. Since the polygon must return to the same point, the parity must be the same as the initial one, thus, we must make an even number of parity changes. That means there is an even number of odd edges, or in other words, the sum of all the edges must be even, so we're done.

- 1 point for determining that edge length is even if the difference in x-coordinates and the difference in y coordinates is even.
- 1 point for determining that edge length is odd if exactly one of the above differences is odd.
- 1 point for showing that after all the parity changes between edges are made, the parity must return to what it was originally (if they say something like since a path traveling around all the edges returns to the same point, thus the number of changes of parity is the same will suffice for this part)
- 2 points for deducing that the number of parity changes must be even
- 1 point for showing that an even number of parity changes (and thus an even number of odd edges) implies that the perimeter is even.

P2. Given an integer $n \geq 2$, the graph G is defined by:
- Vertices of G are represented by binary strings of length n
- Two vertices a, b are connected by an edge if and only if they differ in exactly 2 places

Let S be a subset of the vertices of G, and let S′ be the set of edges between vertices in S and vertices not in S. Show that if |S| (the size of S) $\leq 2^{n-2}$, then S′ $\geq |S|$.

Solution: We induct on n.

Notation: Let V represent the vertices in G throughout this solution.

Base Case: $n = 2$. S can have either 0 or 1 vertex. If S has 0 vertices then $|S| = 0$, trivial. If $|S| = 1$, since all vertices are connected to another vertex, then $|S'| = 1$ as well.

Induction Hypothesis: Assume the claim holds for n. Define subsets S_0 and S_1 which are disjoint subsets of S such that all vertices in S_0 have binary representation beginning in 0, and all vertices in S_1 have binary representation beginning in 1. We proceed to prove that the claim holds for $n + 1$.

Case 1: $|S_0| \leq 2^{n-2}$, and $|S_1| \leq 2^{n-2}$.

Then by applying the induction hypothesis to both subsets, we see that the number of edges that cross between vertices in S and vertices not in S in each of the sets V_0 (all vertices with binary representation beginning in 0) and V_1 (all vertices with binary representation beginning in 1) summed together is at least $|S_0| + |S_1| = |S|$.

Case 2: WLOG, $|S_0| > 2^{(n-2)}$, so then $|S_1| < 2^{(n-2)}$ (because their sum can't be grater than $2^{(n-1)}$):

The induction hypothesis still applies to S_1, and we know that $|S_1'| \geq |S_1|$. We can get a bijection between the elements in V_0 and V_1 by switching the first and last digits in the binary representation of an element in either V_0 or V_1 (ex, 0...1 would become 1...0, and 1...1 would become 0...0). Therefore, we know that there are at least $|S_0| - |S_1|$ edges connecting S_0 with $V_1 - S_1$, and adding these, we see that the number of edges will be at least $|S_0|$. Now we go back to the edges which connect vertices in S_1 with vertices in V_1 but not S_1. These edges represent switching the digit in place i, and the one in place j, with place $i, j < n+1$ (because the first digit needs to still be a 1). Now, note that switching the first digit and the ith digit of a vertex v in V_1 but not S_1 which is the endpoint of one of these edges gives you a vertex in V_0. If the edge's other endpoint is in S_0, then we can add to our count for S'. However, if the edge's other endpoint doesn't end in S_0, this is a problem. However, switching the first and jth digits of the vertex gives you the same vertex in S_1 that was connected to the vertex in V_1 but not S_1, which still allows us to add to our count of edges. Thus, we get at least $|S_1|$ more edges, and have $|S_0| + |S_1| = |S|$ edges total, which completes our proof!

- 1 point for proving the base case correct.
- 1 point for introducing S_0 and S_1, and stating correct induction hypothesis.
- 1 point for proving Case 1.
- 1 point for applying induction hypothesis to S_1 in Case 2.
- 1 point for proving that the number of edges between S_0 and V_1 not including S_1 is at least $|S_0| - |S_1|$.
- 1 point for completing the count to at least $|S|$.