QUESTIONS

1. For the team, power, and tournament rounds, BMT divided up the teams into 14 rooms. You sign up to proctor all 3 rounds, but you cannot proctor in the same room more than once. How many ways can you be assigned for rooms for the 3 rounds?

2. Find the number of 5-digit n, s.t. every digit of n is either 0, 1, 3, or 4, and n is divisible by 15.

3. The Professor chooses to assign homework problems from a set of problems labeled 1 to 100, inclusive. He will not assign two problems whose numbers share a common factor greater than 1. If the Professor chooses to assign the maximum number of homework problems possible, how many different combinations of problems can he assign?

4. What is the sum of the first 31 integers that can be written as a sum of distinct powers of 3?

5. Alice, Bob, and Chris each roll 4 dice. Each only knows the result of their own roll. Alice claims that there are at least 5 multiples of 3 among the dice rolled. Bob has 1 six and no threes, and knows that Alice wouldn't claim such a thing unless he had at least 2 multiples of 3. Bob can call Alice a liar, or claim that there are at least 6 multiples of 3, but Chris says that he will immediately call Bob a liar if he makes this claim. Bob wins if he calls Alice a liar and there aren't at least 5 multiples of 3, or if he claims there are at least 6 multiples of 3, and there are. What is the probability that Bob loses no matter what he does?

6. Pick a 3-digit number abc, which contains no 0's. The probability that this is a *winning number* is $\frac{1}{a} * \frac{1}{b} * \frac{1}{c}$. However, the BMT problem writer tries to balance out the chances for the numbers in the following ways:

   - For the lowest digit n in the number, he rolls an n-sided die for each time that the digit appears, and gives the number 0 probability of winning if an n is rolled.
   - For the largest digit m in the number, he rolls an m-sided die once and scales the probability of winning by that die roll.

   If you choose optimally, what is the probability that your number is a winning number?

7. For a positive integer n, let $\phi(n)$ denote the number of positive integers between 1 and n, inclusive, which are relatively prime to n. We say that a positive integer k is total if $k = \frac{n}{\phi(n)}$, for some positive integer n. Find all total numbers.

8. Suppose that positive integers $a_1, a_2, \ldots, a_{2014}$ (not necessarily distinct) satisfy the condition that: $\frac{a_1}{a_2}, \frac{a_2}{a_3}, \ldots, \frac{a_{2013}}{a_{2014}}$ are pairwise distinct. What is the minimal possible number of distinct numbers in $\{a_1, a_2, \ldots, a_{2014}\}$?

9. Leo and Paul are at the Berkeley BART station and are racing to San Francisco. Leo is planning to take the line that takes him directly to SF, and because he has terrible BART luck, his train will arrive in some integer number of minutes, with probability $\frac{i}{210}$ for $1 \leq i \leq 20$ at any given minute. Paul will take a second line, whose trains always arrive before Leo's train, with uniform probability. However, Paul must also make a transfer to a 3rd line, whose trains arrive with uniform probability between 0 and 10 minutes after Paul reaches the transfer station. What is the probability that Leo gets to SF before Paul does?

10. Let f be a function on $(1, \ldots, n)$ that generates a permutation of $(1, \ldots, n)$. We call a *fixed point* of f any element in the original permutation such that the element's position is not changed when the permutation is applied. Given that n is a multiple of 4, g is a permutation whose fixed points are $(1, \ldots, \frac{n}{2})$, and h is a permutation whose fixed points consist of every element in an even-numbered position. What is the expected number of fixed points in $h(g(1, 2, \ldots, 104))$?

PROOFS

1. Let a simple polygon be defined as a polygon in which no consecutive sides are parallel and no two non-consecutive sides share a common point. Given that all vertices of a simple polygon P are lattice points (in a Cartesian coordinate system, each vertex has integer coordinates), and each side of P has integer length, prove that the perimeter must be even.

2. Given an integer $n \geq 2$, the graph G is defined by:
- Vertices of G are represented by binary strings of length n
- Two vertices a, b are connected by an edge if and only if they differ in exactly 2 places
Let S be a subset of the vertices of G, and let S$'$ be the set of edges between vertices in S and vertices not in S. Show that if |S| (the size of S) $\leq 2^{n-2}$, then S$' \geq |S|$.