1. $x = 0, \log_2 3.$

2. $\dfrac{7-\sqrt{17}}{2}$

3. $\dfrac{1}{5}$

4. -24

5. $\dfrac{1}{2}$

6. $\dfrac{19}{12}.$

7. 289.

8. $\dfrac{8e-2}{4e-e^2}$

9. $\dfrac{3}{2}$

10. $(-10^8 - 50, 100, 10^8 - 50)$

P1. Suppose that a, b, c, d are non-negative real numbers such that $a^2 + b^2 + c^2 + d^2 = 2$ and $ab + bc + cd + da = 1$. Find the maximum value of $a + b + c + d$ and determine all equality cases.

Solution: By AM-GM, $2ac \le a^2 + c^2$ and $2bd \le b^2 + d^2$. Then

$$(a+b+c+d)^2 = a^2+b^2+c^2+d^2+2(ab+bc+cd+da)+2ac+2bd \le 4+(a^2+c^2)+(b^2+d^2) = 6.$$

Thus $a + b + c + d \le \sqrt{6}$. The maximum will occur when $a = c$, $b = d$. Plugging these values back into the given equations, we see $a^2 + b^2 = 1$, $4ab = 1$. The solutions to this are $a, b = \dfrac{\sqrt{6} \pm \sqrt{2}}{4}$. The maximum then occurs when $(a, b, c, d) = (a, b, a, b)$ where $(a, b) = (\dfrac{\sqrt{6}+\sqrt{2}}{4}, \dfrac{\sqrt{6}-\sqrt{2}}{4})$, or $(\dfrac{\sqrt{6}-\sqrt{2}}{4}, \dfrac{\sqrt{6}+\sqrt{2}}{4})$.

- 1 point for stating maximum value of $a + b + c + d$.
- 1 point for stating equality cases. Accept equivalent forms ($a, b = \dfrac{\sqrt{2 \pm \sqrt{3}}}{2}, \dfrac{\sqrt{3} \pm 1}{2\sqrt{2}}$, etc.)
- 2 points for proving maximum.
- 2 points for complete solution.

P2. Define $\eta(f)$ to be the number of roots that are repeated of the complex-valued polynomial f (e.g., $\eta((x-1)^3 \cdot (x+1)^4) = 5$). Prove that for nonconstant, relatively prime $f, g \in \mathbb{C}[x]$,

$$\eta(f) + \eta(g) + \eta(f+g) < \deg f + \deg g.$$

Solution: Note $\eta(f) = \deg(\gcd(f, f'))$. Note $f'g - g'f = f'(f+g) - (f+g)'f$, and since $\gcd(f, f') \mid f$ and $\gcd(f, f') \mid f'$, we conclude $\gcd(f, f') \mid f'g - g'f$, $\gcd(g, g') \mid f'g - g'f$, and $\gcd(f+g, f'+g') \mid f'g - g'f$. Since f, g, and $f + g$ are relatively prime, we conclude

$$\begin{aligned} \gcd(f, f') \cdot \gcd(g, g') \cdot \gcd(f+g, f'+g') &\mid f'g - g'f \\ \deg(\gcd(f, f')) + \deg(\gcd(g, g')) + \deg(\gcd(f+g, f'+g')) &\le \deg(f) + \deg(g) - 1 \\ \eta(f) + \eta(g) + \eta(f+g) &\le \deg(f) + \deg(g) - 1. \end{aligned}$$

- 1 point for realizing $\eta(f) = \deg(\gcd(f, f'))$
- 1 point for showing at least one of $\gcd(f, f') \mid f'g - g'f$, $\gcd(g, g') \mid f'g - g'f$, or $\gcd(f+g, f'+g') \mid f'g - g'f$
- 2 points for realizing $\gcd(f, f') \cdot \gcd(g, g') \cdot \gcd(f+g, f'+g') \mid f'g - g'f$
- 2 points for full solution

In the case of an alternate proof, full points should be given for a rigorous proof, five points should be given for slight errors in computation, four points should be given for easily fixable errors in rigor, two points should be given for solutions with serious flaws, and one point for moderate progress.